Lesson 1: Investigating Properties of Dilations

Common Core Georgia Performance Standards
- MCC9–12.G.SRT.1a
- MCC9–12.G.SRT.1b

Essential Questions
1. How are the preimage and image similar in dilations?
2. How are the preimage and image different in dilations?
3. When are dilations used in the real world?

WORDS TO KNOW
- **center of dilation**: a point through which a dilation takes place; all the points of a dilated figure are stretched or compressed through this point
- **collinear points**: points that lie on the same line
- **compression**: a transformation in which a figure becomes smaller; compressions may be horizontal (affecting only horizontal lengths), vertical (affecting only vertical lengths), or both
- **congruency transformation**: a transformation in which a geometric figure moves but keeps the same size and shape; a dilation where the scale factor is equal to 1
- **corresponding sides**: sides of two figures that lie in the same position relative to the figure. In transformations, the corresponding sides are the preimage and image sides, so $AB$ and $A'B'$ are corresponding sides and so on.
- **dilation**: a transformation in which a figure is either enlarged or reduced by a scale factor in relation to a center point
- **enlargement**: a dilation of a figure where the scale factor is greater than 1
- **non-rigid motion**: a transformation done to a figure that changes the figure’s shape and/or size
- **reduction**: a dilation where the scale factor is between 0 and 1
- **rigid motion**: a transformation done to a figure that maintains the figure’s shape and size or its segment lengths and angle measures
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scale factor  a multiple of the lengths of the sides from one figure to the transformed figure. If the scale factor is larger than 1, then the figure is enlarged. If the scale factor is between 0 and 1, then the figure is reduced.

stretch  a transformation in which a figure becomes larger; stretches may be horizontal (affecting only horizontal lengths), vertical (affecting only vertical lengths), or both

Recommended Resources

  http://www.walch.com/rr/00017
  This interactive website gives a series of problems and scores them immediately. If the user submits a wrong answer, a description and process for arriving at the correct answer are provided. These problems start with a graphed figure. Users are asked to input the coordinates of the dilated figure given a center and scale factor.

  http://www.walch.com/rr/00018
  This interactive website gives a series of problems and scores them immediately. If the user submits a wrong answer, a description and process for arriving at the correct answer are provided. These problems start with a graphed figure. Users are asked to draw a dilation of the figure on the screen using a point that can be dragged, given a center and scale factor.

  http://www.walch.com/rr/00019
  This interactive website gives a series of problems and scores them immediately. If the user submits a wrong answer, a description and process for arriving at the correct answer are provided. These problems start with a graphed preimage and image. Users are required to choose whether the figure is an enlargement or a reduction. Other problems ask users to enter the scale factor.

  http://www.walch.com/rr/00020
  This website gives a brief explanation of the properties of dilations and how to perform them. The site also contains an interactive applet with which users can select a shape, a center point, and a scale factor. The computer then generates the dilated image. After users explore the applet, they may answer eight multiple-choice questions in order to check understanding.
A college has a wide-open space in the center of the campus. Landscape architects have proposed gardens at the four corners of the open space. Each garden would be in the shape of a triangle. The first garden is pictured below.

1. Perform a transformation of the first garden to create a second garden at the opposite vertex of the open space. The opposite vertex lies in Quadrant IV. Write the coordinates of the new garden and state the transformation you used.

2. Is the transformation you performed a rigid motion? Explain.
Lesson 1.1.1: Investigating Properties of Parallelism and the Center

Common Core Georgia Performance Standard

MCC9–12.G.SRT.1a

Warm-Up 1.1.1 Debrief

A college has a wide-open space in the center of the campus. Landscape architects have proposed gardens at the four corners of the open space. Each garden would be in the shape of a triangle. The first garden is pictured below.

1. Perform a transformation of the first garden to create a second garden at the opposite vertex of the open space. The opposite vertex lies in Quadrant IV. Write the coordinates of the new garden and state the transformation you used.

There are two possibilities.

The first is to reflect the garden over the line $y = x$. 
Lesson 1: Investigating Properties of Dilations

Instruction

\[ r_{y=x} \begin{pmatrix} x, y \end{pmatrix} = \begin{pmatrix} y, x \end{pmatrix} \]

\[ A^\prime = r_{y=x}(A) = r_{y=x}(-7, 3) = (3, -7) \]

\[ B^\prime = r_{y=x}(B) = r_{y=x}(-3, 7) = (7, -3) \]

\[ C^\prime = r_{y=x}(C) = r_{y=x}(-3, 3) = (3, -3) \]

This results in the following graph:

The second possibility is to rotate the garden \(180^\circ\) counterclockwise about the origin.

\[ R_{180}(x, y) = (-x, -y) \]

\[ A^\prime = R_{180}(A) = R_{180}(-7, 3) = (7, -3) \]

\[ B^\prime = R_{180}(B) = R_{180}(-3, 7) = (3, -7) \]

\[ C^\prime = R_{180}(C) = R_{180}(-3, 3) = (3, -3) \]
This results in the following graph:

Observe that the shapes of the graphs for both possibilities are identical.

2. Is the transformation you performed a rigid motion? Explain.

Reflections and rotations are rigid motions. They preserve angle and length measures. The result of either transformation is a congruent figure.

**Connection to the Lesson**
- Students will extend previous understandings of rigid motions to include non-rigid motions.
- Students will work with vertices on the coordinate plane.
Prerequisite Skills

This lesson requires the use of the following skills:

- operating with fractions, including complex fractions
- operating with decimals
- calculating slope
- determining parallel lines

Introduction

Think about resizing a window on your computer screen. You can stretch it vertically, horizontally, or at the corner so that it stretches both horizontally and vertically at the same time. These are non-rigid motions. **Non-rigid motions** are transformations done to a figure that change the figure’s shape and/or size. These are in contrast to **rigid motions**, which are transformations to a figure that maintain the figure’s shape and size, or its segment lengths and angle measures.

Specifically, we are going to study non-rigid motions of dilations. **Dilations** are transformations in which a figure is either enlarged or reduced by a scale factor in relation to a center point.

Key Concepts

- Dilations require a center of dilation and a scale factor.
- The **center of dilation** is the point about which all points are stretched or compressed.
- The **scale factor** of a figure is a multiple of the lengths of the sides from one figure to the transformed figure.
- Side lengths are changed according to the scale factor, \( k \).
- The scale factor can be found by finding the distances of the sides of the preimage in relation to the image.
- Use a ratio of corresponding sides to find the scale factor: 
  \[
  \frac{\text{length of image side}}{\text{length of preimage side}} = \text{scale factor}
  \]
- The scale factor, \( k \), takes a point \( P \) and moves it along a line in relation to the center so that \( k \cdot CP = CP' \).
If the scale factor is greater than 1, the figure is stretched or made larger and is called an **enlargement**. (A transformation in which a figure becomes larger is also called a **stretch**.)

If the scale factor is between 0 and 1, the figure is compressed or made smaller and is called a **reduction**. (A transformation in which a figure becomes smaller is also called a **compression**.)

If the scale factor is equal to 1, the preimage and image are congruent. This is called a **congruency transformation**.

Angle measures are preserved in dilations.

The orientation is also preserved.

The sides of the preimage are parallel to the corresponding sides of the image.

The **corresponding sides** are the sides of two figures that lie in the same position relative to the figures.

In transformations, the corresponding sides are the preimage and image sides, so $\overline{AB}$ and $\overline{A'B'}$ are corresponding sides and so on.

The notation of a dilation in the coordinate plane is given by $D_k(x, y) = (kx, ky)$. The scale factor is multiplied by each coordinate in the ordered pair.

The center of dilation is usually the origin, $(0, 0)$. 
If a segment of the figure being dilated passes through the center of dilation, then the image segment will lie on the same line as the preimage segment. All other segments of the image will be parallel to the corresponding preimage segments.

The corresponding points in the preimage and image are **collinear points**, meaning they lie on the same line, with the center of dilation.

### Properties of Dilations

1. Shape, orientation, and angles are preserved.
2. All sides change by a single scale factor, $k$.
3. The corresponding preimage and image sides are parallel.
4. The corresponding points of the figure are collinear with the center of dilation.

### Common Errors/Misconceptions

- Forgetting to check the ratio of all sides from the image to the preimage in determining if a dilation has occurred
- Inconsistently setting up the ratio of the side lengths
- Confusing enlargements with reductions and vice versa
Guided Practice 1.1.1

Example 1

Is the following transformation a dilation? Justify your answer using the properties of dilations.

1. Verify that shape, orientation, and angles have been preserved from the preimage to the image.

   Both figures are triangles in the same orientation.
   \[ \angle D \cong \angle D' \]
   \[ \angle E \cong \angle E' \]
   \[ \angle F \cong \angle F' \]

   The angle measures have been preserved.
2. Verify that the corresponding sides are parallel.

\[ m_{DE} = \frac{\Delta y}{\Delta x} = \frac{(2-1)}{(-2-2)} = -\frac{1}{4} \quad \text{and} \quad m_{D' E'} = \frac{\Delta y'}{\Delta x'} = \frac{(4-2)}{(-4-4)} = -\frac{1}{4} \]

therefore, \( DE \parallel D'E' \).

By inspection, \( EF \parallel E'F' \) because both lines are vertical; therefore, they have the same slope and are parallel.

\[ m_{EF} = \frac{\Delta y}{\Delta x} = \frac{[2-(-2)]}{(-2-2)} = -1 \quad \text{and} \quad m_{E'F'} = \frac{\Delta y'}{\Delta x'} = \frac{[4-(-4)]}{(-4-4)} = -1 \]

therefore, \( DF \parallel D'F' \). In fact, these two segments, \( DF \) and \( D'F' \), lie on the same line.

All corresponding sides are parallel.

3. Verify that the distances of the corresponding sides have changed by a common scale factor, \( k \).

We could calculate the distances of each side, but that would take a lot of time. Instead, examine the coordinates and determine if the coordinates of the vertices have changed by a common scale factor.

The notation of a dilation in the coordinate plane is given by \( D_k(x, y) = (kx, ky) \).

Divide the coordinates of each vertex to determine if there is a common scale factor.

\[
\begin{align*}
D(-2, 2) &\rightarrow D'(-4, 4) \\
x_{D'} &= -\frac{4}{-2} = 2; \quad y_{D'} = \frac{4}{2} = 2 \\
x_D &= -2; \quad y_D = 2
\end{align*}
\]

\[
\begin{align*}
E(2, 1) &\rightarrow E'(4, 2) \\
x_{E'} &= \frac{4}{2} = 2; \quad y_{E'} = \frac{2}{1} = 2 \\
x_E &= 2; \quad y_E = 1
\end{align*}
\]

\[
\begin{align*}
F(2, -2) &\rightarrow F'(4, -4) \\
x_{F'} &= \frac{4}{2} = 2; \quad y_{F'} = \frac{-4}{-2} = 2 \\
x_F &= 2; \quad y_F = -2
\end{align*}
\]

Each vertex’s preimage coordinate is multiplied by 2 to create the corresponding image vertex. Therefore, the common scale factor is \( k = 2 \).
4. Verify that corresponding vertices are collinear with the center of dilation, \( C \).

A straight line can be drawn connecting the center with the corresponding vertices. This means that the corresponding vertices are collinear with the center of dilation.

5. Draw conclusions.

The transformation is a dilation because the shape, orientation, and angle measures have been preserved. Additionally, the size has changed by a scale factor of 2. All corresponding sides are parallel, and the corresponding vertices are collinear with the center of dilation.
Example 2

Is the following transformation a dilation? Justify your answer using the properties of dilations.

1. Verify that shape, orientation, and angles have been preserved from the preimage to the image.

The preimage and image are both rectangles with the same orientation. The angle measures have been preserved since all angles are right angles.
2. Verify that the corresponding sides are parallel.
   \( \overline{TU'} \) is on the same line as \( \overline{TU} \); therefore, \( \overline{TU} \parallel \overline{TU'} \).
   \( \overline{CV'} \) is on the same line as \( \overline{CV} \); therefore, \( \overline{CV} \parallel \overline{CV'} \).
   By inspection, \( \overline{UV} \) and \( \overline{U'V'} \) are vertical; therefore, \( \overline{UV} \parallel \overline{U'V'} \).
   \( \overline{TC} \) remains unchanged from the preimage to the image.
   All corresponding sides are parallel.

3. Verify that the distances of the corresponding sides have changed by a common scale factor, \( k \).
   Since the segments of the figure are on a coordinate plane and are either horizontal or vertical, find the distance by counting.
   
   \[
   \begin{align*}
   &\text{In } \square TUVC : & \text{In } \square T'U'V'C : \\
   &TU = VC = 9 & T'U' = V'C = 6 \\
   &UV = CT = 5 & U'V' = CT = 5 \\
   \end{align*}
   \]
   The formula for calculating the scale factor is:
   \[
   \text{scale factor} = \frac{\text{length of image side}}{\text{length of preimage side}}
   \]
   Start with the horizontal sides of the rectangle.
   \[
   \begin{align*}
   &\frac{T'U'}{TU} = \frac{6}{9} = \frac{2}{3} & \frac{V'C}{VC} = \frac{6}{9} = \frac{2}{3} \\
   \end{align*}
   \]
   Both corresponding horizontal sides have a scale factor of \( \frac{2}{3} \).
   
   Next, calculate the scale factor of the vertical sides.
   \[
   \begin{align*}
   &\frac{U'V'}{UV} = \frac{5}{5} = 1 & \frac{CT}{CT} = \frac{5}{5} = 1 \\
   \end{align*}
   \]
   Both corresponding vertical sides have a scale factor of 1.
4. Draw conclusions.

The vertical corresponding sides have a scale factor that is not consistent with the scale factor of $\frac{2}{3}$ for the horizontal sides. Since all corresponding sides do not have the same common scale factor, the transformation is NOT a dilation.

Example 3

The following transformation represents a dilation. What is the scale factor? Does this indicate enlargement, reduction, or congruence?

1. Determine the scale factor.

Start with the ratio of one set of corresponding sides.

$$\text{scale factor} = \frac{\text{length of image side}}{\text{length of preimage side}}$$

$$\frac{A'B'}{AB} = \frac{2.5}{10} = \frac{1}{4}$$

The scale factor appears to be $\frac{1}{4}$. 

[Diagram of triangle A'B'C' with sides labeled 2.5, 3.75, and 3, and triangle ABC with sides labeled 12, 10, and 15]
2. Verify that the other sides maintain the same scale factor.

\[
\frac{B'C'}{BC} = \frac{3.75}{15} = \frac{1}{4} \quad \text{and} \quad \frac{C'A'}{CA} = \frac{3}{12} = \frac{1}{4}.
\]

Therefore,

\[
\frac{A'B'}{AB} = \frac{B'C'}{BC} = \frac{C'A'}{CA} = \frac{1}{4}
\]

and the scale factor, \( k \), is \( \frac{1}{4} \).

3. Determine the type of dilation that has occurred.

If \( k > 1 \), then the dilation is an enlargement.

If \( 0 < k < 1 \), then the dilation is a reduction.

If \( k = 1 \), then the dilation is a congruency transformation.

Since \( k = \frac{1}{4} \), \( k \) is between 0 and 1, or \( 0 < k < 1 \).

The dilation is a reduction.
Problem-Based Task 1.1.1: Prettying Up the Pentagon

The Pentagon, diagrammed below, is one of the world’s largest office buildings. The outside walls are each 921 feet long and are a dilation of the inner walls through the center of the courtyard. The courtyard is the area inside the inner wall. Since the courtyard is surrounded by the inner walls, each side of the courtyard is the same length as the inner walls. The dashed lines represent a walkway that borders a garden. The walkway is a dilation of the inner wall of the office building.

A team of landscapers has been hired to update the courtyard. The landscapers need to know the perimeter of the walkway in order to install some temporary fencing while the courtyard is redone. What is the perimeter of the walkway if the dilation from the inner wall to the walkway has a scale factor of 0.25? What relationship does the scale factor have to the perimeters of the figures?
Problem-Based Task 1.1.1: Prettying Up the Pentagon

Coaching

a. What is the length of one side of the inner wall (the preimage)?

b. What is the scale factor used to dilate the inner wall to determine the walkway?

c. How do you calculate the length of each side of the walkway in the pentagonal figure in the courtyard?

d. What is the length of each side of the walkway?

e. How many sides are there in a pentagon?

f. What is the perimeter of the walkway?

g. What is the perimeter of the inner wall?

h. What is the scale factor of the perimeter of the inner wall to the perimeter of the walkway?

i. How do the scale factors of the lengths of each side of the inner wall and the walkway compare to the perimeters of the inner wall and the walkway?

j. What is the scale factor of the inner wall to the outer wall?

k. What is the perimeter of the outer wall?

l. What is the scale factor of the perimeter of the inner wall to the perimeter of the outer wall?

m. How do the scale factors of the individual side lengths of the inner and outer walls compare to the perimeters of the inner and outer walls?

n. What can you conclude about the scale factors of perimeters of dilated figures?
Problem-Based Task 1.1.1: Prettying Up the Pentagon

Coaching Sample Responses

a. What is the length of one side of the inner wall (the preimage)?
   264 feet

b. What is the scale factor used to dilate the inner wall to determine the walkway?
   0.25

c. How do you calculate the length of each side of the walkway in the pentagonal figure in the courtyard?
   Multiply the preimage side by the scale factor: 264(0.25).

  d. What is the length of each side of the walkway?
    264(0.25) = 66
    The length of each side of the walkway in the pentagonal figure is 66 feet.

e. How many sides are there in a pentagon?
   A pentagon has 5 sides.

f. What is the perimeter of the walkway?
   5(66) = 330
   The perimeter of the walkway is 330 feet.

g. What is the perimeter of the inner wall?
   5(264) = 1320
   The perimeter of the inner wall is 1,320 feet.

h. What is the scale factor of the perimeter of the inner wall to the perimeter of the walkway?
   \[
   \frac{\text{perimeter of walkway}}{\text{perimeter of inner wall}} = \frac{330}{1320} = 0.25
   \]
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i. How do the scale factors of the lengths of each side of the inner wall and the walkway compare to the perimeters of the inner wall and the walkway?

The scale factor of each wall is the same as the scale factor of the perimeters.

j. What is the scale factor of the inner wall to the outer wall?

\[
\frac{\text{length of inner wall}}{\text{length of outer wall}} = \frac{264}{921} \approx 0.29
\]

k. What is the perimeter of the outer wall?

\[5(921) = 4605\]

The perimeter of the outer wall is 4,605 feet.

l. What is the scale factor of the perimeter of the inner wall to the perimeter of the outer wall?

\[
\frac{\text{perimeter of inner wall}}{\text{perimeter of outer wall}} = \frac{1320}{4605} = 0.29
\]

m. How do the scale factors of the individual side lengths of the inner and outer walls compare to the perimeters of the inner and outer walls?

The scale factors are the same.

n. What can you conclude about the scale factors of perimeters of dilated figures?

If a figure is dilated, the perimeter of the preimage to the image has the same scale factor as the dilation.

**Recommended Closure Activity**

Select one or more of the essential questions for a class discussion or as a journal entry prompt.
Practice 1.1.1: Investigating Properties of Parallelism and the Center

Determine whether each of the following transformations represents a dilation. Justify your answer using the properties of dilations.

1. Compare polygon $CMNOP$ to polygon $CM'N'O'P'$.

2. Compare $\triangle TUV$ to $\triangle T'U'V'$.
3. Compare $\triangle QRC$ to $\triangle QR'C$.

4. Compare $\square CMNO$ to $\square CM'N'O$. 

continued
For problems 5 and 6, the following transformations represent dilations. Determine the scale factor and whether the dilation is an enlargement, a reduction, or a congruency transformation.

5.

6.

Use the given information in each problem that follows to answer the questions.

7. A right triangle has the following side lengths: \( AB = 13 \), \( BC = 12 \), and \( CA = 5 \). The triangle is dilated so that the image has side lengths \( A'B' = \frac{26}{5} \), \( B'C' = \frac{24}{5} \), and \( C'A' = 2 \). What is the scale factor? Does this represent an enlargement, a reduction, or a congruency transformation?
8. Derald is building a playhouse for his daughter. He wants the playhouse to look just like the family’s home, so he’s using the drawings from his house plans to create the plans for the playhouse. The diagram below shows part of a scale drawing of a roof truss used in the house. What is the scale factor of the roof truss from the house drawing to the playhouse drawing?
9. On Board, a luggage manufacturer, has had great success with a certain model of carry-on luggage. Feedback suggests that customers would prefer that the company sell different sizes of luggage with the same design as the carry-on. The graph below represents the top view of the original carry-on model and a proposed larger version of the same luggage. Does the new piece of luggage represent a dilation of the original piece of luggage? Why or why not?

10. A university wants to put in a courtyard for a new building. The courtyard is bounded by the coordinates $P(-4, 0)$, $Q(-2, -6)$, $R(6, 2)$, and $S(0, 4)$. The landscape architects created a dilation of the space through the center $C(0, 0)$ to outline the garden. The garden is bounded by the points $P'(−2.4, 0)$, $Q'(−1.2, −3.6)$, $R'(3.6, 1.2)$, and $S'(0, 2.4)$. What is the scale factor? Does this represent an enlargement, a reduction, or a congruency transformation?
Lesson 1.1.2: Investigating Scale Factors

Warm-Up 1.1.2

Hideki is babysitting his little sister and takes her to the park. While pushing her on the swing, he notices that if he pushes her so that she swings as high as the top of his head and then lets her go without pushing her, she only swings as high as his shoulders.

1. If Hideki is 6 feet tall and the height at his shoulders is 5.1 feet, what is the scale factor of the change in height of his sister’s swing?

2. Rewrite this change as a percentage.

3. What is the reciprocal of this change in height?

4. Rewrite this reciprocal as a percentage.

5. What might it mean in the context of the problem if you applied the reciprocal of the change to the height when Hideki’s sister is swinging as high as the top of his head?
Lesson 1.1.2: Investigating Scale Factors

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Warm-Up 1.1.2 Debrief

Hideki is babysitting his little sister and takes her to the park. While pushing her on the swing he notices that if he pushes her so that she swings as high as the top of his head and then lets her go without pushing her, she only swings as high as his shoulders.

1. If Hideki is 6 feet tall and the height at his shoulders is 5.1 feet, what is the scale factor of the change in height of his sister’s swing?
   \[
   \frac{\text{height after 1 swing}}{\text{initial height}} = \frac{5.1}{6} = 0.85
   \]

2. Rewrite this change as a percentage.
   \[0.85(100) = 85\%\]

3. What is the reciprocal of this change in height?
   The reciprocal of \(\frac{5.1}{6}\) is \(\frac{6}{5.1}\).

4. Rewrite this reciprocal as a percentage.
   \[
   \frac{6}{5.1} = 1.18
   \]
   \[1.18(100) = 118\%
   \]

5. What might it mean in the context of the problem if you applied the reciprocal of the change to the height when Hideki’s sister is swinging as high as the top of his head?
   Hideki might have pushed his sister again to cause her to swing higher.

Connection to the Lesson

- Students will need to be able to convert decimals to percents.
- Students can draw on the connection between scale factors and percent change in terms of when the dilation is an enlargement or a reduction. Students can use this connection to predict the effects of dilating a figure given a scale factor and center.
**Prerequisite Skills**

This lesson requires the use of the following skills:

- operating with fractions, decimals, and percents
- converting among fractions, decimals, and percents

**Introduction**

A figure is dilated if the preimage can be mapped to the image using a scale factor through a center point, usually the origin. You have been determining if figures have been dilated, but how do you create a dilation? If the dilation is centered about the origin, use the scale factor and multiply each coordinate in the figure by that scale factor. If a distance is given, multiply the distance by the scale factor.

**Key Concepts**

- The notation is as follows: \( D_1(x, y) = (kx, ky) \).
- Multiply each coordinate of the figure by the scale factor when the center is at \((0, 0)\).

- The lengths of each side in a figure also are multiplied by the scale factor.
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Instruction

• If you know the lengths of the preimage figure and the scale factor, you can calculate the lengths of the image by multiplying the preimage lengths by the scale factor.

• Remember that the dilation is an enlargement if $k > 1$, a reduction if $0 < k < 1$, and a congruency transformation if $k = 1$.

Common Errors/Misconceptions

• not applying the scale factor to both the $x$- and $y$-coordinates in the point
• improperly converting the decimal from a percentage
• missing the connection between the scale factor and the ratio of the image lengths to the preimage lengths
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Guided Practice 1.1.2

Example 1
If \( \overline{AB} \) has a length of 3 units and is dilated by a scale factor of 2.25, what is the length of \( \overline{A'B'} \)? Does this represent an enlargement or a reduction?

1. To determine the length of \( \overline{A'B'} \), multiply the scale factor by the length of the segment.
   \[ AB = 3; \ k = 2.25 \]
   \[ A'B' = k \cdot AB \]
   \[ A'B' = 2.25 \cdot 3 = 6.75 \]
   \( \overline{A'B'} \) is 6.75 units long.

2. Determine the type of dilation.
   Since the scale factor is greater than 1, the dilation is an enlargement.

Example 2
A triangle has vertices \( G(2, -3), H(-6, 2) \), and \( J(0, 4) \). If the triangle is dilated by a scale factor of 0.5 through center \( C(0, 0) \), what are the image vertices? Draw the preimage and image on the coordinate plane.

1. Start with one vertex and multiply each coordinate by the scale factor, \( k \).
   \[ G' = D_{0.5}[G(2, -3)] = D_{0.5}(0.5 \cdot 2, 0.5 \cdot -3) = (1, -1.5) \]

2. Repeat the process with another vertex. Multiply each coordinate of the vertex by the scale factor.
   \[ H' = D_{0.5}[H(-6, 2)] = D_{0.5}(0.5 \cdot -6, 0.5 \cdot 2) = (-3, 1) \]
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3. Repeat the process for the last vertex. Multiply each coordinate of the vertex by the scale factor.

\[ J' = D_{0.5}(J(0, 4)) = D_{0.5}(0.5 \cdot 0, 0.5 \cdot 4) = (0, 2) \]

4. List the image vertices.

- \( G' (1, -1.5) \)
- \( H' (-3, 1) \)
- \( J' (0, 2) \)

5. Draw the preimage and image on the coordinate plane.
Example 3

What are the side lengths of $\triangle D'E'F'$ with a scale factor of 2.5 given the preimage and image below and the information that $DE = 1$, $EF = 9.2$, and $FD = 8.6$?

1. Choose a side to start with and multiply the scale factor $(k)$ by that side length.
   
   $DE = 1; \ k = 2.5$
   
   $D'E' = k \cdot DE$
   
   $D'E' = 2.5 \cdot 1 = 2.5$

2. Choose a second side and multiply the scale factor by that side length.
   
   $EF = 9.2; \ k = 2.5$
   
   $E'F' = k \cdot EF$
   
   $E'F' = 2.5 \cdot 9.2 = 23$
3. Choose the last side and multiply the scale factor by that side length.

\[ FD = 8.6; \ k = 2.5 \]

\[ F'D' = k \cdot FD \]

\[ F'D' = 2.5 \cdot 8.6 = 21.5 \]

4. Label the figure with the side lengths.
Problem-Based Task 1.1.2: The Bigger Picture

A photographer wants to enlarge a 5 × 7 picture to an 8 × 10. However, she wants to preserve the image as it appears in the 5 × 7 without distorting the picture. Distortions happen when the width and height of the photo are not enlarged at the same scale. How can the photographer dilate a 5 × 7 picture to an 8 × 10 picture without distorting the picture? Describe a process for enlarging the picture so that the image is a dilation of the preimage. Give the coordinates for the image vertices. The preimage is pictured below with the center C (0, 0).
Problem-Based Task 1.1.2: The Bigger Picture

Coaching

a. What is the scale factor of the width from the preimage to the image?

b. What is the scale factor of the height from the preimage to the image?

c. How do these scale factors compare?

d. Which scale factor can you use consistently with the width and the height so that the picture will not be distorted?

e. How can you modify the preimage so that you can use the same scale factor and arrive at an 8 × 10 picture?

f. How can you determine numerically how to modify the picture?

g. What are the coordinates of the modified preimage?

h. What are the coordinates of the image after applying the scale factor?

i. Summarize your procedure and thinking process in dilating the original photograph.
Problem-Based Task 1.1.2: The Bigger Picture

Coaching Sample Responses

a. What is the scale factor of the width from the preimage to the image?
\[
\frac{\text{width of image}}{\text{width of preimage}} = \frac{8}{5} = 1.6
\]

b. What is the scale factor of the height from the preimage to the image?
\[
\frac{\text{height of image}}{\text{height of preimage}} = \frac{10}{7} \approx 1.429
\]

c. How do these scale factors compare?

The scale factor of the height is smaller than the scale factor for the width.

d. Which scale factor can you use consistently with the width and the height so that the picture will not be distorted?

Examine what happens when you apply the scale factor backward. In other words, find the reciprocal of each scale factor.

The scale factor for the width = 8/5. The reciprocal is 5/8 or 0.625.

The scale factor for the height = 10/7. The reciprocal is 7/10 or 0.7.

Use the smaller reciprocal scale factor since you cannot generate more of an original picture. This means when you go from the smaller preimage to the image, you will use the reciprocal of 5/8, which is 8/5, the scale factor of the width.

e. How can you modify the preimage so that you can use the same scale factor and arrive at an 8 \times 10 picture?

Crop or trim the picture width so that the preimage width becomes smaller.

f. How can you determine numerically how to modify the picture?

To determine by how much to trim the picture, set up a proportion with the unknown being the height of the original picture. Since we are using one scale factor for both dimensions, the second ratio is the ratio of the width of the image to the width of the preimage.
The original height is 7 inches. The picture needs to be trimmed by $7 - 6.25$ inches, or 0.75 inches. The image could be trimmed 0.75 inches at the bottom or the top or by a combination to avoid losing any of the important aspects of the picture.

g. What are the coordinates of the modified preimage?

Assuming equal trimming of the top and bottom of the picture, the amount to trim at each end is $0.75/2 = 0.375$ inches. For negative coordinates, you must add 0.375 to show the trimming.

$D: 3.5 - 0.375 = 3.125$
$E: 3.5 - 0.375 = 3.125$
$F: -3.5 + 0.375 = -3.125$
$G: -3.5 + 0.375 = -3.125$

The coordinates of the modified preimage are as follows:

$D(-2.5, 3.5) \rightarrow D_m(-2.5, 3.125)$
$E(2.5, 3.5) \rightarrow E_m(2.5, 3.125)$
$F(2.5, -3.5) \rightarrow F_m(2.5, -3.125)$
$G(-2.5, -3.5) \rightarrow G_m(-2.5, -3.125)$

h. What are the coordinates of the image after applying the scale factor?

Apply the scale factor of $8/5$ or 1.6 to each coordinate in the modified preimage.

$D_m(-2.5, 3.125) \rightarrow D'_m(-4, 5)$
$E_m(2.5, 3.125) \rightarrow E'_m(4, 5)$
$F_m(2.5, -3.125) \rightarrow F'_m(4, -5)$
$G_m(-2.5, -3.125) \rightarrow G'_m(-4, -5)$
i. Summarize your procedure and thinking process in dilating the original photograph.

First, determine the scale factor to use by calculating the scale factor for each dimension of width and height. Use the scale factor that will allow you to trim the photograph rather than adding onto it. This means you use the scale factor with the smaller reciprocal because you will be going backward to determine how much to cut from the picture. The scale factor to use is the scale factor of the width, which is 8/5 or 1.6. The unknown is how long the image should be before dilation. You know that you want to end up with 10 inches in height. Apply the scale factor to 10 inches and the result is 6.25 inches. The picture height is 7 inches.

To modify the height of the picture, trim 0.75 inches from the picture. This can be done equally from the top and bottom (0.375 inches from the top and 0.375 inches from the bottom), from the top only, from the bottom only, or some combination of the top and bottom that is not equal. For simplicity, trim 0.375 inches from the top and the bottom. On the graph, this means moving each of the $y$-coordinates 0.375 units toward 0 parallel to the $y$-axis. The $x$-coordinates will remain the same because the width is not changing. Now, you apply the scale factor of 8/5 to each coordinate in all the vertices. The resulting image is a dilation of the trimmed picture.

**Recommended Closure Activity**

Select one or more of the essential questions for a class discussion or as a journal entry prompt.
Practice 1.1.2: Investigating Scale Factors

Determine the lengths of the dilated segments given the preimage length and the scale factor.

1. $\overline{AB}$ is 2.25 units long and the segment is dilated by a scale factor of $k = 3.2$.

2. $\overline{GH}$ is 15.3 units long and is dilated by a scale factor of $k = \frac{2}{3}$.

3. $\overline{ST}$ is 20.5 units long and is dilated by a scale factor of $k = 0.6$.

4. $\overline{DE}$ is 30 units long and is dilated by a scale factor of $k = \frac{2}{3}$.

Determine the image vertices of each dilation given a center and scale factor.

5. $\triangle HJK$ has the following vertices: $H(–7, –3)$, $J(–5, –6)$, and $K(–6, –8)$. What are the vertices under a dilation with a center at $(0, 0)$ and a scale factor of 3?

6. $\triangle PQR$ has the following vertices: $P(–6, 4)$, $Q(5, 9)$, and $R(–3, –6)$. What are the vertices under a dilation with a center at $(0, 0)$ and a scale factor of $\frac{1}{2}$?

7. $\triangle MNO$ has the following vertices: $M(–5, 8)$, $N(7, –3)$, and $O(–10, –4)$. What are the vertices under a dilation with a center at $(0, 0)$ and a scale factor of 75%?

8. $\triangle ABD$ has the following vertices: $A(6, 5)$, $B(2, 2)$, and $D(–3, 4)$. What are the vertices under a dilation with a center at $(0, 0)$ and a scale factor of 140%?

9. $\triangle DEF$ has the following vertices: $D(3, 2)$, $E(6, 2)$, and $F(–1.5, 4)$. What are the final vertices after 2 successive dilations with a center at $(0, 0)$ and a scale factor of 2? What is the scale factor from $\triangle DEF$ to $\triangle D''E''F''$?

10. Miguel’s family is renovating their kitchen. Miguel is comparing the floor plan for the old kitchen to the floor plan for the new kitchen. According to the floor plan for the new kitchen, the center island is going to be enlarged by a scale factor of 1.5. If in the old floor plan, the vertices of the original countertop are $S(–3, 2)$, $T(3, 2)$, $U(3, –2)$, and $V(–3, –2)$, what are the vertices of the new countertop? By what factor is the new countertop longer than the original? Assume each unit of the coordinate plane represents 1 foot.
Lesson 2: Constructing Lines, Segments, and Angles

Common Core Georgia Performance Standard
MCC9–12.G.CO.12

Essential Questions
1. What is the difference between sketching geometric figures, drawing geometric figures, and constructing geometric figures?
2. What tools are used with geometric constructions and why?
3. How can you justify a construction was done correctly?

WORDS TO KNOW

**altitude**
the perpendicular line from a vertex of a figure to its opposite side; height

**angle**
two rays or line segments sharing a common endpoint; the symbol used is \( \angle \)

**bisect**
to cut in half

**compass**
an instrument for creating circles or transferring measurements that consists of two pointed branches joined at the top by a pivot

**congruent**
having the same shape, size, or angle

**construct**
to create a precise geometric representation using a straightedge along with either patty paper (tracing paper), a compass, or a reflecting device

**construction**
a precise representation of a figure using a straightedge and a compass, patty paper and a straightedge, or a reflecting device and a straightedge

**drawing**
a precise representation of a figure, created with measurement tools such as a protractor and a ruler

**endpoint**
either of two points that mark the ends of a line segment; a point that marks the end of a ray

**equidistant**
the same distance from a reference point
<table>
<thead>
<tr>
<th>Term</th>
<th>Definition</th>
</tr>
</thead>
<tbody>
<tr>
<td>line</td>
<td>the set of points between two points ( P ) and ( Q ) and the infinite number of points that continue beyond those points</td>
</tr>
<tr>
<td>median of a triangle</td>
<td>the segment joining the vertex to the midpoint of the opposite side</td>
</tr>
<tr>
<td>midpoint</td>
<td>a point on a line segment that divides the segment into two equal parts</td>
</tr>
<tr>
<td>midsegment</td>
<td>a line segment joining the midpoints of two sides of a figure</td>
</tr>
<tr>
<td>parallel lines</td>
<td>lines that either do not share any points and never intersect, or share all points</td>
</tr>
<tr>
<td>perpendicular bisector</td>
<td>a perpendicular line constructed through the midpoint of a segment</td>
</tr>
<tr>
<td>perpendicular lines</td>
<td>two lines that intersect at a right angle (90°)</td>
</tr>
<tr>
<td>ray</td>
<td>a line with only one endpoint</td>
</tr>
<tr>
<td>segment</td>
<td>a part of a line that is noted by two endpoints</td>
</tr>
<tr>
<td>sketch</td>
<td>a quickly done representation of a figure; a rough approximation of a figure</td>
</tr>
<tr>
<td>straightedge</td>
<td>a bar or strip of wood, plastic, or metal having at least one long edge of reliable straightness</td>
</tr>
</tbody>
</table>
Recommended Resources

  This site provides step-by-step instructions for bisecting an angle with a compass and straightedge.

- Math Open Reference. “Copying a line segment.”
  This site provides step-by-step instructions for copying a line segment with a compass and straightedge.

- Math Open Reference. “Copying an angle.”
  This site provides step-by-step instructions for copying an angle with a compass and straightedge.

- Math Open Reference. “Perpendicular at a point on a line.”
  [http://www.walch.com/rr/00004](http://www.walch.com/rr/00004)
  This site provides step-by-step instructions for constructing a perpendicular line at a point on the given line.

- Math Open Reference. “Perpendicular bisector of a line segment.”
  This site provides step-by-step instructions for constructing a perpendicular bisector of a line segment using a compass and straightedge.

- Math Open Reference. “Perpendicular to a line from an external point.”
  This site provides step-by-step instructions for constructing a perpendicular line from a point not on the given line.
Maryellen is caring for a pet goat. To keep the goat from roaming the neighborhood, Maryellen ties the goat to a leash staked in the yard. Maryellen learned quickly that goats enjoy chewing grass, so she must change the location of the stake periodically.

The original location of the stake is shown below on the left with the leash stretched out. Maryellen moved the stake to a new location, shown below on the right. Use the diagrams to solve the problems that follow.

1. Using only a compass, determine if there will be an overlap in the area the goat will chew.

2. Maryellen wants to keep both her goat and her yard healthy. Describe a process for determining where Maryellen should place each new stake in order to prevent the goat from overgrazing on the same patch of grass.
Lesson 1.2.1: Copying Segments and Angles

Common Core Georgia Performance Standard

MCC9–12.G.CO.12

Warm-Up 1.2.1 Debrief

Maryellen is caring for a pet goat. To keep the goat from roaming the neighborhood, Maryellen ties the goat to a leash staked in the yard. Maryellen learned quickly that goats enjoy chewing grass, so she must change the location of the stake periodically.

The original location of the stake is shown below on the left with the leash stretched out. Maryellen moved the stake to a new location, shown below on the right. Use the diagrams to solve the problems that follow.

Original stake location  New stake location

1. Using only a compass, determine if there will be an overlap in the area the goat will chew.

First, put the sharp point of the compass at the original stake location.
Open the compass so the pencil point reaches the end of the leash.
Draw a circle, keeping the sharp point at the original stake location.
Without adjusting the opening of the compass, place the sharp point of the compass on the new stake location.

Draw a circle, keeping the sharp point at the new stake location.

The circles do not overlap; therefore, the areas chewed by the goat will not overlap.

2. Maryellen wants to keep both her goat and her yard healthy. Describe a process for determining where Maryellen should place each new stake in order to prevent the goat from overgrazing on the same patch of grass.

Using the leash that has already been staked, Maryellen could mark the area the goat can chew by pulling the leash tight and walking the circle.
Maryellen could then place one end of the leash on the outside edge of the circle, untie the leash from the stake, and stretch the leash out in the opposite direction from where the goat was staked.

She could then place the stake in the ground.

The leash acts like a compass for determining the circle.

Maryellen could continue doing this to minimize the amount of overlap, and ensure that she uses the space in her yard the most effectively.

**Connection to the Lesson**

- Students will practice using a compass to make circles.
- It is important that students are capable of creating circles effortlessly before proceeding.
- Students will determine lengths with tools other than rulers.
Introduction

Two basic instruments used in geometry are the straightedge and the compass. A straightedge is a bar or strip of wood, plastic, or metal that has at least one long edge of reliable straightness, similar to a ruler, but without any measurement markings. A compass is an instrument for creating circles or transferring measurements. It consists of two pointed branches joined at the top by a pivot. It is believed that during early geometry, all geometric figures were created using just a straightedge and a compass. Though technology and computers abound today to help us make sense of geometry problems, the straightedge and compass are still widely used to construct figures, or create precise geometric representations. Constructions allow you to draw accurate segments and angles, segment and angle bisectors, and parallel and perpendicular lines.

Key Concepts

- A geometric figure precisely created using only a straightedge and compass is called a construction.

- A straightedge can be used with patty paper (tracing paper) or a reflecting device to create precise representations.

- Constructions are different from drawings or sketches.

- A drawing is a precise representation of a figure, created with measurement tools such as a protractor and a ruler.

- A sketch is a quickly done representation of a figure or a rough approximation of a figure.

- When constructing figures, it is very important not to erase your markings.

- Markings show that your figure was constructed and not measured and drawn.

- An endpoint is either of two points that mark the ends of a line, or the point that marks the end of a ray.

- A line segment is a part of a line that is noted by two endpoints.
• An angle is formed when two rays or line segments share a common endpoint.
• A constructed figure and the original figure are congruent; they have the same shape, size, or angle.
• Follow the steps outlined below and on the next page to copy a segment and an angle.

### Copying a Segment Using a Compass

1. To copy $\overline{AB}$, first make an endpoint on your paper. Label the endpoint $C$.
2. Put the sharp point of your compass on endpoint $A$. Open the compass until the pencil end touches endpoint $B$.
3. Without changing your compass setting, put the sharp point of your compass on endpoint $C$. Make a large arc.
4. Use your straightedge to connect endpoint $C$ to any point on your arc.
5. Label the point of intersection of the arc and your segment $D$.

Do not erase any of your markings.

$\overline{AB}$ is congruent to $\overline{CD}$.

### Copying a Segment Using Patty Paper

1. To copy $\overline{AB}$, place your sheet of patty paper over the segment.
2. Mark the endpoints of the segment on the patty paper. Label the endpoints $C$ and $D$.
3. Use your straightedge to connect points $C$ and $D$.

$\overline{AB}$ is congruent to $\overline{CD}$. 
Copying an Angle Using a Compass

1. To copy \( \angle A \), first make a point to represent the vertex \( A \) on your paper. Label the vertex \( E \).
2. From point \( E \), draw a ray of any length. This will be one side of the constructed angle.
3. Put the sharp point of the compass on vertex \( A \) of the original angle. Set the compass to any width that will cross both sides of the original angle.
4. Draw an arc across both sides of \( \angle A \). Label where the arc intersects the angle as points \( B \) and \( C \).
5. Without changing the compass setting, put the sharp point of the compass on point \( E \). Draw a large arc that intersects the ray. Label the point of intersection as \( F \).
6. Put the sharp point of the compass on point \( B \) of the original angle and set the width of the compass so it touches point \( C \).
7. Without changing the compass setting, put the sharp point of the compass on point \( F \) and make an arc that intersects the arc in step 5. Label the point of intersection as \( D \).
8. Draw a ray from point \( E \) to point \( D \).

Do not erase any of your markings.

\( \angle A \) is congruent to \( \angle E \).

Copying an Angle Using Patty Paper

1. To copy \( \angle A \), place your sheet of patty paper over the angle.
2. Mark the vertex of the angle. Label the vertex \( E \).
3. Use your straightedge to trace each side of \( \angle A \).

\( \angle A \) is congruent to \( \angle E \).

Common Errors/Misconceptions

- inappropriately changing the compass setting
- moving the patty paper before completing the construction
- attempting to measure lengths and angles with rulers and protractors
Guided Practice 1.2.1

Example 1

Copy the following segment using only a compass and a straightedge.

1. Make an endpoint on your paper. Label the endpoint $P$.

   \[
   \text{Original segment} \quad \text{Construction}
   \]

   \[
   \begin{array}{c}
   M
   \end{array}
   \quad
   \begin{array}{c}
   N
   \end{array}
   \quad
   \begin{array}{c}
   P
   \end{array}
   \]

2. Put the sharp point of your compass on endpoint $M$. Open the compass until the pencil end touches endpoint $N$.

   \[
   \text{Original segment} \quad \text{Construction}
   \]

   \[
   \begin{array}{c}
   M
   \end{array}
   \quad
   \begin{array}{c}
   N
   \end{array}
   \quad
   \begin{array}{c}
   P
   \end{array}
   \]
3. Without changing your compass setting, put the sharp point of your compass on endpoint $P$. Make a large arc.

4. Use your straightedge to connect endpoint $P$ to any point on your arc.

5. Label the point of intersection of the arc and your segment $Q$.

Do not erase any of your markings. $\overline{MN}$ is congruent to $\overline{PQ}$. 

\[ \overline{MN} \cong \overline{PQ} \]
Example 2

Copy the following angle using only a compass and a straightedge.

1. Make a point to represent vertex \( J \). Label the vertex \( R \).

2. From point \( R \), draw a ray of any length. This will be one side of the constructed angle.
3. Put the sharp point of the compass on vertex \( J \) of the original angle. Set the compass to any width that will cross both sides of the original angle.

![Original angle](image1)

![Construction](image2)

4. Draw an arc across both sides of \( \angle J \). Label where the arc intersects the angle as points \( K \) and \( L \).

![Original angle](image3)

![Construction](image4)

5. Without changing the compass setting, put the sharp point of the compass on point \( R \). Draw a large arc that intersects the ray. Label the point of intersection as \( S \).

![Original angle](image5)

![Construction](image6)
6. Put the sharp point of the compass on point \( L \) of the original angle and set the width of the compass so it touches point \( K \).

7. Without changing the compass setting, put the sharp point of the compass on point \( S \) and make an arc that intersects the arc you drew in step 5. Label the point of intersection as \( T \).

8. Draw a ray from point \( R \) to point \( T \).

\[ \angle J \text{ is congruent to } \angle R. \]
Example 3

Use the given line segment to construct a new line segment with length $2AB$.

1. Use your straightedge to draw a long ray. Label the endpoint $C$.

2. Put the sharp point of your compass on endpoint $A$ of the original segment. Open the compass until the pencil end touches $B$.

3. Without changing your compass setting, put the sharp point of your compass on $C$ and make a large arc that intersects your ray.

4. Mark the point of intersection as point $D$. 
5. Without changing your compass setting, put the sharp point of your compass on $D$ and make a large arc that intersects your ray.

6. Mark the point of intersection as point $E$.

Do not erase any of your markings.

$CE = 2AB$
Example 4

Use the given angle to construct a new angle equal to $\angle A + \angle A$.

1. Follow the steps from Example 2 to copy $\angle A$. Label the vertex of the copied angle $G$.

2. Put the sharp point of the compass on vertex $A$ of the original angle. Set the compass to any width that will cross both sides of the original angle.

3. Draw an arc across both sides of $\angle A$. Label where the arc intersects the angle as points $B$ and $C$. 

Image of a diagram showing the construction steps.
4. Without changing the compass setting, put the sharp point of the compass on $G$. Draw a large arc that intersects one side of your newly constructed angle. Label the point of intersection $H$.

5. Put the sharp point of the compass on $C$ of the original angle and set the width of the compass so it touches $B$.

6. Without changing the compass setting, put the sharp point of the compass on point $H$ and make an arc that intersects the arc created in step 4. Label the point of intersection as $J$. 
7. Draw a ray from point G to point J.

Do not erase any of your markings.

\[ \angle G = \angle A + \angle A \]

Example 5
Use the given segments to construct a new segment equal to \( AB - CD \).

1. Draw a ray longer than that of \( AB \). Label the endpoint \( M \).

2. Follow the steps from Example 3 to copy \( AB \) onto the ray. Label the second endpoint \( P \).
3. Put the sharp point of the compass on endpoint $M$ of the ray. Copy segment $CD$ onto the same ray. Label the endpoint $N$.

Do not erase any of your markings.

$$NP = AB - CD$$

Angles can be subtracted in the same way.
Problem-Based Task 1.2.1: How Many Triangles?

How many non-congruent triangles can be constructed using the two sides and angle given? Use your construction tools to show all possible triangles.

What additional information could you provide to be sure that only one triangle is constructed?
Problem-Based Task 1.2.1: How Many Triangles?

Coaching

a. A triangle is made up of how many sides?

b. A triangle is made up of how many angles?

c. Are there enough sides and angles to create a triangle?

d. Is it possible to create at least one triangle using the given segments and angle?

e. Using a compass and a straightedge, what is the method for copying an angle?

f. What is the method for copying a segment?

g. Construct and label a triangle using the given segments and angle.

h. Is it possible to rearrange the placement of the sides and create a second triangle that is not congruent to the first? Explain your reasoning.

i. If possible, construct and label a second triangle not congruent to the first using the given segments and angle.

j. Is it possible to rearrange the placement of the sides and create a third triangle that is not congruent to the first or second? Explain your reasoning.

k. If possible, construct and label a third triangle not congruent to the first or second using the given segments and angle.

l. Is it possible to rearrange the placement of the sides and create a fourth triangle that is not congruent to the first, second, or third? Explain your reasoning.

m. If possible, construct and label a fourth triangle not congruent to the first, second, or third using the given segments and angle.

n. Is it possible to rearrange the placement of the sides and create a fifth triangle that is not congruent to the first, second, third, or fourth? Explain your reasoning.

o. If possible, construct and label a fifth triangle not congruent to the first, second, third, or fourth using the given segments and angle.

p. How many non-congruent triangles were you able to create using the given segments and angle?

q. What additional information could you provide to be sure that only one triangle could be created?
Problem-Based Task 1.2.1: How Many Triangles?

Coaching Sample Responses

a. A triangle is made up of how many sides?
   A triangle is made up of 3 sides.

b. A triangle is made up of how many angles?
   A triangle is made up of 3 angles.

c. Are there enough sides and angles to create a triangle?
   Two segments are given; each one can be used as a side.
   One angle is given.
   The remaining side and angles can be created once the given information is used.

d. Is it possible to create at least one triangle using the given segments and angle?
   A triangle can be created if the given angle and side lengths are constructed first.
   Then the sides can be connected to create the third side and remaining angles.

e. Using a compass and a straightedge, what is the method for copying an angle?
   Make a point to represent the vertex. Label the vertex $A$.
   From point $A$, draw a ray of any length. This will be one side of the constructed angle.
   Put the sharp point of the compass on the vertex of the given angle.
   Set the compass to any width that will cross both sides of the given angle.
   Draw an arc across both sides of the given angle.
   Label where the arc intersects the angle as points $C$ and $D$.
   Without changing the compass setting, put the sharp point of the compass on point $A$. Draw a large arc that intersects the ray. Label the point of intersection as $S$.
   Put the sharp point of the compass on point $C$ and set the width of the compass so it touches point $D$.
   Without changing the compass setting, put the sharp point of the compass on point $S$ and make an arc that intersects the arc.
   Label the point of intersection as $T$.
   Draw a ray from point $A$ to point $T$. 
f. What is the method for copying a segment?
   Make an endpoint on your paper. Label the endpoint $P$.
   Put the sharp point of your compass on one endpoint of the given segment.
   Open the compass until the pencil end touches the second endpoint of the given segment.
   Without changing your compass setting, put the sharp point of your compass on endpoint $P$.
   Make a large arc.
   Label any point on the large arc as $Q$.
   Use your straightedge to connect endpoint $P$ to point $Q$.

f. Construct and label a triangle using the given segments and angle.
   Check students’ papers for accuracy.

h. Is it possible to rearrange the placement of the sides and create a second triangle that is not congruent to the first? Explain your reasoning.
   Yes, because the placement of the sides is not given in the information provided. Students can rearrange the placement of each segment.

i. If possible, construct and label a second triangle not congruent to the first using the given segments and angle.
   Check students’ papers for accuracy.

j. Is it possible to rearrange the placement of the sides and create a third triangle that is not congruent to the first or second? Explain your reasoning.
   Yes, because the placement of the sides is not given in the information provided. Students can rearrange the placement of each segment.

k. If possible, construct and label a third triangle not congruent to the first or second using the given segments and angle.
   Check students’ papers for accuracy.

l. Is it possible to rearrange the placement of the sides and create a fourth triangle that is not congruent to the first, second, or third? Explain your reasoning.
   Yes, because the placement of the sides is not given in the information provided. Students can rearrange the placement of each segment.

m. If possible, construct and label a fourth triangle not congruent to the first, second, or third using the given segments and angle.
   Check students’ papers for accuracy.
n. Is it possible to rearrange the placement of the sides and create a fifth triangle that is not congruent to the first, second, third, or fourth? Explain your reasoning.

No, it is not possible to create a fifth triangle that is not congruent to the first, second, third, or fourth triangles. All combinations of sides have been exhausted.

o. If possible, construct and label a fifth triangle not congruent to the first, second, third, or fourth using the given segments and angle.

Since this is not possible, students’ papers should not have a fifth triangle.

p. How many non-congruent triangles were you able to create using the given segments and angle?

Four non-congruent triangles can be created using the given segments and angle. The four non-congruent triangles include the following (scaled):

```
<table>
<thead>
<tr>
<th>Angle A</th>
<th>Side 1</th>
<th>Side 2</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
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<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
```

![Diagram of non-congruent triangles](image-url)
q. What additional information could you provide to be sure that only one triangle could be created?

To be sure that only one triangle can be drawn, more specific information would need to be provided.

If the angle has vertex $A$, Side 1 has endpoints $A$ and $B$, and Side 2 has endpoints $A$ and $C$, then only one congruent triangle can be drawn.

**Recommended Closure Activity**

Select one or more of the essential questions for a class discussion or as a journal entry prompt.
Practice 1.2.1: Copying Segments and Angles

Copy the following segments using a straightedge and a compass.

1. [Diagram of segment AB]
2. [Diagram of segment CD]
3. [Diagram of segment EF]

Use the line segments in problems 1–3 to construct the line segments described in problems 4 and 5.

4. $EF - AB$
5. $2AB + CD$

Copy the following angles using a straightedge and a compass.

6. [Diagram of angle G]
7. [Diagram of angle H]
8. [Diagram of angle J]

Use the angles from problems 6–8 to construct the angles described in problems 9 and 10.

9. $\angle J - \angle G$
10. $\angle G + \angle H$
Henri is designing a set for the upcoming school play. The play opens with the main character in front of the clock tower at 1:00 P.M. Henri easily constructs the hands of the clock for the first scene, but needs help with the placement of the hands for the second scene. The second scene takes place 2 hours later in front of the same tower.

1. Use a compass and a straightedge to construct the hands of the clock for the second scene. Be sure to use construction methods previously learned.

2. If the measure of the angle of the hands for the first scene is 30°, what is the measure of the angle of the hands for the second scene?
Lesson 1.2.2: Bisecting Segments and Angles

Common Core Georgia Performance Standard

MCC9–12.G.CO.12

Warm-Up 1.2.2 Debrief

Henri is designing a set for the upcoming school play. The play opens with the main character in front of the clock tower at 1:00 P.M. Henri easily constructs the hands of the clock for the first scene, but needs help with the placement of the hands for the second scene. The second scene takes place 2 hours later in front of the same tower.

1. Use a compass and a straightedge to construct the hands of the clock for the second scene. Be sure to use construction methods previously learned.

Use the vertex of the original angle as the vertex for the new angle of the clock hands.

Copy the original angle, using the lower side of the angle as the starting ray (the hour hand).

Copy the original angle again to construct the position for the second scene, with the ray representing the hour hand at the 3:00 position.

The problem scenario specified that the second scene takes place 2 hours later, so the original angle ray representing the minute hand should stay in place, at the 12:00 position. The second angle ray representing the hour hand is at 3:00. Therefore, the angle that represents the time of the second scene is encompassed by the rays pointing at 12:00 and 3:00.
Check students’ papers for accuracy.

2. If the measure of the angle of the hands for the first scene is 30°, what is the measure of the angle of the hands for the second scene?
   
   The original position is 30°.
   
   The second scene is 2 hours later, meaning the angle opened an additional 60° (30 • 2 = 60). The resulting angle is 90° (30 + 60 = 90).

**Connection to the Lesson**

- Students continue to practice copying angles and using construction tools.
- In this lesson, students will use similar construction skills to partition angles.
Introduction

Segments and angles are often described with measurements. Segments have lengths that can be measured with a ruler. Angles have measures that can be determined by a protractor. It is possible to determine the midpoint of a segment. The midpoint is a point on the segment that divides it into two equal parts. When drawing the midpoint, you can measure the length of the segment and divide the length in half. When constructing the midpoint, you cannot use a ruler, but you can use a compass and a straightedge (or patty paper and a straightedge) to determine the midpoint of the segment. This procedure is called bisecting a segment. To bisect means to cut in half. It is also possible to bisect an angle, or cut an angle in half using the same construction tools. A midsegment is created when two midpoints of a figure are connected. A triangle has three midsegments.

Key Concepts

Bisecting a Segment

- A segment bisector cuts a segment in half.
- Each half of the segment measures exactly the same length.
- A point, line, ray, or segment can bisect a segment.
- A point on the bisector is equidistant, or is the same distance, from either endpoint of the segment.
- The point where the segment is bisected is called the midpoint of the segment.
Bisecting a Segment Using a Compass

1. To bisect $\overline{AB}$, put the sharp point of your compass on endpoint $A$. Open the compass wider than half the distance of $\overline{AB}$.
2. Make a large arc intersecting $\overline{AB}$.
3. Without changing your compass setting, put the sharp point of the compass on endpoint $B$. Make a second large arc. It is important that the arcs intersect each other in two places.
4. Use your straightedge to connect the points of intersection of the arcs.
5. Label the midpoint of the segment $C$.

Do not erase any of your markings.

$\overline{AC}$ is congruent to $\overline{BC}$.

Bisecting a Segment Using Patty Paper

1. Use a straightedge to construct $\overline{AB}$ on patty paper.
2. Fold the patty paper so point $A$ meets point $B$. Be sure to crease the paper.
3. Unfold the patty paper.
4. Use your straightedge to mark the midpoint of $\overline{AB}$.
5. Label the midpoint of the segment $C$.

$\overline{AC}$ is congruent to $\overline{BC}$.

Bisecting an Angle

- An angle bisector cuts an angle in half.
- Each half of the angle has exactly the same measure.
- A line or ray can bisect an angle.
- A point on the bisector is equidistant, or is the same distance, from either side of the angle.
**Bisecting an Angle Using a Compass**

1. To bisect $\angle A$, put the sharp point of the compass on the vertex of the angle.
2. Draw a large arc that passes through each side of the angle.
3. Label where the arc intersects the angle as points $B$ and $C$.
4. Put the sharp point of the compass on point $B$. Open the compass wider than half the distance from $B$ to $C$.
5. Make a large arc.
6. Without changing the compass setting, put the sharp point of the compass on $C$.
7. Make a second large arc. It is important that the arcs intersect each other in two places.
8. Use your straightedge to create a ray connecting the points of intersection of the arcs with the vertex of the angle, $A$.
9. Label a point, $D$, on the ray.

Do not erase any of your markings.

$\angle CAD$ is congruent to $\angle BAD$.

**Bisecting an Angle Using Patty Paper**

1. Use a straightedge to construct $\angle A$ on patty paper.
2. Fold the patty paper so the sides of $\angle A$ line up. Be sure to crease the paper.
3. Unfold the patty paper.
4. Use your straightedge to mark the crease line with a ray.
5. Label a point, $D$, on the ray.

$\angle CAD$ is congruent to $\angle BAD$.

**Common Errors/Misconceptions**

- inappropriately changing the compass setting
- moving the patty paper before completing the construction
- not creating large enough arcs to find the point of intersection
- attempting to measure lengths and angles with rulers and protractors
Guided Practice 1.2.2

Example 1

Use a compass and straightedge to find the midpoint of $\overline{CD}$. Label the midpoint of the segment $M$.

1. Copy the segment and label it $\overline{CD}$.

2. Make a large arc intersecting $\overline{CD}$.

   Put the sharp point of your compass on endpoint $C$. Open the compass wider than half the distance of $\overline{CD}$. Draw the arc.
3. Make a second large arc.

Without changing your compass setting, put the sharp point of the compass on endpoint $D$, then make the second arc.

It is important that the arcs intersect each other in two places.

4. Connect the points of intersection of the arcs.

Use your straightedge to connect the points of intersection.
5. Label the midpoint of the segment \( M \).

Do not erase any of your markings.

\( CM \) is congruent to \( MD \).

Example 2

Construct a segment whose measure is \( \frac{1}{4} \) the length of \( PQ \).

1. Copy the segment and label it \( PQ \).
2. Make a large arc intersecting $PQ$.

Put the sharp point of your compass on endpoint $P$. Open the compass wider than half the distance of $PQ$. Draw the arc.

3. Make a second large arc.

Without changing your compass setting, put the sharp point of the compass on endpoint $Q$, then make the second arc.

It is important that the arcs intersect each other in two places.
4. Connect the points of intersection of the arcs.

Use your straightedge to connect the points of intersection. Label the midpoint of the segment $M$.

\[ PM \text{ is congruent to } MQ. \]
\[ PM \text{ and } MQ \text{ are both } \frac{1}{2} \text{ the length of } PQ. \]

5. Find the midpoint of $PM$.

Make a large arc intersecting $PM$.

Put the sharp point of your compass on endpoint $P$. Open the compass wider than half the distance of $PM$. Draw the arc.
6. Make a second large arc.

Without changing your compass setting, put the sharp point of the compass on endpoint $M$, and then draw the second arc. It is important that the arcs intersect each other in two places.

7. Connect the points of intersection of the arcs.

Use your straightedge to connect the points of intersection. Label the midpoint of the smaller segment $N$.

Do not erase any of your markings.

$\overline{PN}$ is congruent to $\overline{NM}$.

$\frac{1}{4}$ the length of $\overline{PQ}$.
Example 3

Use a compass and a straightedge to bisect an angle.

1. Draw an angle and label the vertex \( \angle J \).

2. Make a large arc intersecting the sides of \( \angle J \).

   Put the sharp point of the compass on the vertex of the angle and swing the compass so that it passes through each side of the angle.

   Label where the arc intersects the angle as points \( L \) and \( M \).
3. Find a point that is equidistant from both sides of $\angle J$.
   
   Put the sharp point of the compass on point $L$.
   
   Open the compass wider than half the distance from $L$ to $M$.
   
   Make an arc beyond the arc you made for points $L$ and $M$.

Without changing the compass setting, put the sharp point of the compass on $M$.

Make a second arc that crosses the arc you just made. It is important that the arcs intersect each other.

Label the point of intersection $N$. 

4. Draw the angle bisector.

Use your straightedge to create a ray connecting the point $N$ with the vertex of the angle, $J$.

Do not erase any of your markings.

$\angle LJM$ is congruent to $\angle NJM$.

**Example 4**

Construct an angle whose measure is $\frac{3}{4}$ the measure of $\angle S$.

1. Copy the angle and label the vertex $S$. 
2. Make a large arc intersecting the sides of $\angle S$.
   Put the sharp point of the compass on the vertex of the angle and swing the compass so that it passes through each side of the angle. Label where the arc intersects the angle as points $T$ and $U$.

3. Find a point that is equidistant from both sides of $\angle S$.
   Put the sharp point of the compass on point $T$.
   Open the compass wider than half the distance from $T$ to $U$.
   Make an arc beyond the arc you made for points $T$ and $U$.

   Without changing the compass setting, put the sharp point of the compass on $U$.
   Make a second arc that crosses the arc you just made. It is important that the arcs intersect each other.
   Label the point of intersection $W$. 
4. Draw the angle bisector.
   Use your straightedge to create a ray connecting the point $W$ with the vertex of the angle, $S$.

\[ \angle TSW \text{ is congruent to } \angle WSU. \]

The measure of $\angle TSW$ is $\frac{1}{2}$ the measure of $\angle S$.

The measure of $\angle WSU$ is $\frac{1}{2}$ the measure of $\angle S$.

5. Find a point that is equidistant from both sides of $\angle WSU$.

Label the intersection of the angle bisector and the initial arc as $X$.

Put the sharp point of the compass on point $X$.

Open the compass wider than half the distance from $X$ to $U$.

Make an arc.
Without changing the compass setting, put the sharp point of the compass on $U$.

Make a second arc. It is important that the arcs intersect each other. Label the point of intersection $Z$.

5. Draw the angle bisector.

Use your straightedge to create a ray connecting point $Z$ with the vertex of the original angle, $S$.

Do not erase any of your markings.

$\angle XSZ$ is congruent to $\angle ZSU$.

$\angle TSZ$ is $\frac{3}{4}$ the measure of $\angle TSU$. 
Problem-Based Task 1.2.2: Triangle Medians

The **median of a triangle** is a line segment joining the vertex of a triangle to the midpoint of the opposite side. What happens when all medians of one triangle are constructed?
Problem-Based Task 1.2.2: Triangle Medians

Coaching

a. What is a midpoint?

b. How many midpoints does a triangle have?

c. How many medians does a triangle have?

Use the triangle below for the following constructions.

\[ \begin{array}{c}
A \\
B \\
C 
\end{array} \]

\[ \overline{AB}, \overline{BC}, \overline{AC} \]

d. What is the process for constructing the midpoint of \( \overline{AB} \)?

e. Construct the midpoint of \( \overline{AB} \).

f. Construct the midpoint of \( \overline{BC} \).

g. Construct the midpoint of \( \overline{AC} \).

h. Construct each median.

i. What happens when all medians of the triangle are constructed?

j. Is this true for all triangles?

k. How can you prove your answer to be correct?
Problem-Based Task 1.2.2: Triangle Medians

Coaching Sample Responses

a. What is a midpoint?
   A midpoint is a point on a line segment that divides the segment into two equal parts.

b. How many midpoints does a triangle have?
   A triangle has 3 midpoints—one on each side.

c. How many medians does a triangle have?
   A triangle has 3 medians—one connecting each vertex to the midpoint of the opposite side.

Use the triangle below for the following constructions.

```
A

B

C
```

d. What is the process for constructing the midpoint of $\overline{AB}$?
   The construction of the midpoint of $\overline{AB}$ is the same as constructing the segment bisector of $\overline{AB}$.
   Make a large arc intersecting $\overline{AB}$.
   Put the sharp point of your compass on endpoint $A$.
   Open the compass wider than half the distance of $\overline{AB}$.
   Make a second large arc.
   Without changing your compass setting, put the sharp point of the compass on endpoint $B$.
   Connect the points of intersection of the arcs.
   Use a straightedge to connect the points of intersection.
   Label the midpoint of the segment.
e. Construct the midpoint of $\overline{AB}$.
    Check students’ papers for accuracy.

f. Construct the midpoint of $\overline{BC}$.
    Check students’ papers for accuracy.

g. Construct the midpoint of $\overline{AC}$.
    Check students’ papers for accuracy.

h. Construct each median.
    Check students’ papers for accuracy.

i. What happens when all medians of the triangle are constructed?
    If constructed correctly, all three medians will intersect at the same point.

j. Is this true for all triangles?
    The three medians of every triangle will always intersect at one point.

k. How can you prove your answer to be correct?
    Draw another triangle and construct each median.
    Compare results with others.

**Recommended Closure Activity**
Select one or more of the essential questions for a class discussion or as a journal entry prompt.
Practice 1.2.2: Bisecting Segments and Angles

Use a compass and straightedge to copy each segment, and then construct the bisector of each segment.

1. \( \overline{AB} \)

2. \( \overline{CD} \)

3. \( \overline{EF} \)

Use a compass and straightedge to construct each segment as specified.

4. Construct a segment whose measure is \( \frac{1}{4} \) the length of \( \overline{AB} \) in problem 1.

5. Construct a segment whose measure is \( \frac{3}{4} \) the length of \( \overline{EF} \) in problem 3.
Use a compass and straightedge to copy each angle, and then construct the bisector of each angle.

6.

7.

8.

Use a compass and straightedge to construct each angle as specified.

9. Construct an angle whose measure is $\frac{1}{4}$ the measure of $\angle G$ in problem 6.

10. Construct an angle whose measure is $\frac{3}{4}$ the measure of $\angle H$ in problem 7.
Lesson 1.2.3: Constructing Perpendicular and Parallel Lines

Warm-Up 1.2.3

Mikhail would like to create a soccer field in the rectangular field behind his house.

1. Mikhail first needs to create the midline of the soccer field. The midline is the line that is created when the midpoint of the longer sides of the field are connected. Construct the midline of the soccer field.

2. Mikhail must also create the center circle of the field. The center circle is located on the midline of the field and is equidistant from both the longer sides. Construct the center circle of the soccer field.
Lesson 1.2.3: Constructing Perpendicular and Parallel Lines

Common Core Georgia Performance Standard

MCC9–12.G.CO.12

Warm-Up 1.2.3 Debrief

Mikhail would like to create a soccer field in the rectangular field behind his house.

1. Mikhail first needs to create the midline of the soccer field. The midline is the line that is created when the midpoints of the longer sides of the field are connected. Construct the midline of the soccer field.

Find the midpoint of each of the longer sides of the field by constructing the segment bisectors. Connect the segment bisectors. See the illustration on the following page.
2. Mikhail must also create the center circle of the field. The center circle is located on the midline of the field and is equidistant from both the longer sides. Construct the center circle of the soccer field.

Find the midpoint of each of the shorter sides of the field by constructing the segment bisectors.

Connect the segment bisectors.

Construct a circle with the center at the intersection of the midline and the centerline. See the illustration on the following page.
Connection to the Lesson

- Students continue to practice constructing segment bisectors.
- In this lesson, students will use similar construction skills to construct perpendicular bisectors and parallel lines.
Prerequisite Skills
This lesson requires the use of the following skills:

- using a compass
- copying angles and segments
- bisecting line segments
- understanding the geometry terms line, segment, ray, and angle

Introduction
Geometry construction tools can also be used to create perpendicular and parallel lines. While performing each construction, it is important to remember that the only tools you are allowed to use are a compass and a straightedge, a reflective device and a straightedge, or patty paper and a straightedge. You may be tempted to measure angles or lengths, but in constructions this is not allowed. You can adjust the opening of your compass to verify that lengths are equal.

Key Concepts
Perpendicular Lines and Bisectors
- **Perpendicular lines** are two lines that intersect at a right angle (90°).
- A perpendicular line can be constructed through the midpoint of a segment. This line is called the **perpendicular bisector** of the line segment.
- It is impossible to create a perpendicular bisector of a line, since a line goes on infinitely in both directions, but similar methods can be used to construct a line perpendicular to a given line.
- It is possible to construct a perpendicular line through a point on the given line as well as through a point not on a given line.
Constructing a Perpendicular Bisector of a Line Segment Using a Compass

1. To construct a perpendicular bisector of $\overline{AB}$, put the sharp point of your compass on endpoint $A$. Open the compass wider than half the distance of $\overline{AB}$.
2. Make a large arc intersecting $\overline{AB}$.
3. Without changing your compass setting, put the sharp point of the compass on endpoint $B$. Make a second large arc. It is important that the arcs intersect each other.
4. Use your straightedge to connect the points of intersection of the arcs.
5. Label the new line $m$.

Do not erase any of your markings.

$\overline{AB}$ is perpendicular to line $m$.

Constructing a Perpendicular Bisector of a Line Segment Using Patty Paper

1. Use a straightedge to construct $\overline{AB}$ onto patty paper.
2. Fold the patty paper so point $A$ meets point $B$. Be sure to crease the paper.
3. Unfold the patty paper.
4. Use your straightedge to mark the creased line.
5. Label the new line $m$.

$\overline{AB}$ is perpendicular to line $m$. 
**Constructing a Perpendicular Line Through a Point on the Given Line Using a Compass**

1. To construct a perpendicular line through the point, \( A \), on a line, put the sharp point of your compass on point \( A \). The opening of the compass does not matter, but try to choose a setting that isn’t so large or so small that it’s difficult to make markings.

2. Make an arc on either side of point \( A \) on the line. Label the points of intersection \( C \) and \( D \).

3. Place the sharp point of the compass on point \( C \). Open the compass so it extends beyond point \( A \).

4. Create an arc on either side of the line.

5. Without changing your compass setting, put the sharp point of the compass on endpoint \( D \). Make a large arc on either side of the line. It is important that the arcs intersect each other.

6. Use your straightedge to connect the points of intersection of the arcs.

7. Label the new line \( m \).

Do not erase any of your markings.

\( CD \) is perpendicular to line \( m \) through point \( A \).

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**Constructing a Perpendicular Line Through a Point on the Given Line Using Patty Paper**

1. Use a straightedge to construct a line, \( \ell \), on the patty paper. Label a point on the line \( A \).

2. Fold the patty paper so the line folds onto itself through point \( A \). Be sure to crease the paper.

3. Unfold the patty paper.

4. Use your straightedge to mark the creased line.

5. Label the new line \( m \).

Line \( \ell \) is perpendicular to line \( m \) through point \( A \).
Instruction

**Constructing a Perpendicular Line Through a Point Not on the Given Line Using a Compass**

1. To construct a perpendicular line through the point, \( G \), not on the given line \( \ell \), put the sharp point of your compass on point \( G \). Open the compass until it extends farther than the given line.

2. Make a large arc that intersects the given line in exactly two places. Label the points of intersection \( C \) and \( D \).

3. Without changing your compass setting, put the sharp point of the compass on point \( C \). Make a second arc below the given line.

4. Without changing your compass setting, put the sharp point of the compass on point \( D \). Make a third arc below the given line. The third arc must intersect the second arc.

5. Label the point of intersection \( E \).

6. Use your straightedge to connect points \( G \) and \( E \). Label the new line \( m \).

Do not erase any of your markings.

Line \( \ell \) is perpendicular to line \( m \) through point \( G \).

**Constructing a Perpendicular Line Through a Point Not on the Given Line Using Patty Paper**

1. Use a straightedge to construct a line, \( \ell \), on the patty paper. Label a point not on the line, \( G \).

2. Fold the patty paper so the line folds onto itself through point \( G \). Be sure to crease the paper.

3. Unfold the patty paper.

4. Use your straightedge to mark the creased line.

5. Label the new line \( m \).

Line \( \ell \) is perpendicular to line \( m \) through point \( G \).

**Parallel Lines**

- **Parallel lines** are lines that either do not share any points and never intersect, or share all points.

- Any two points on one parallel line are equidistant from the other line.
• There are many ways to construct parallel lines.
• One method is to construct two lines that are both perpendicular to the same given line.

**Constructing a Parallel Line Using a Compass**

1. To construct a parallel line through a point, $A$, not on the given line $\ell$, first construct a line perpendicular to $\ell$.
2. Put the sharp point of your compass on point $A$. Open the compass until it extends farther than line $\ell$.
3. Make a large arc that intersects the given line in exactly two places. Label the points of intersection $C$ and $D$.
4. Without changing your compass setting, put the sharp point of the compass on point $C$. Make a second arc below the given line.
5. Without changing your compass setting, put the sharp point of the compass on point $D$. Make a third arc below the given line. The third arc must intersect the second arc.
6. Label the point of intersection $E$.
7. Use your straightedge to connect points $A$ and $E$. Label the new line $m$. Line $m$ is perpendicular to line $\ell$.
8. Construct a second line perpendicular to line $m$.
9. Put the sharp point of your compass on point $A$. Open the compass until it extends farther than line $m$.
10. Make a large arc that intersects line $m$ in exactly two places. Label the points of intersection $F$ and $G$.
11. Without changing your compass setting, put the sharp point of the compass on point $F$. Make a second arc to the right of line $m$.
12. Without changing your compass setting, put the sharp point of the compass on point $G$. Make a third arc to the right of line $m$. The third arc must intersect the second arc.
13. Label the point of intersection $H$.
14. Use your straightedge to connect points $A$ and $H$. Label the new line $n$.

Do not erase any of your markings.

Line $n$ is perpendicular to line $m$.
Line $\ell$ is parallel to line $n$. 
Constructing a Parallel Line Using Patty Paper

1. Use a straightedge to construct line \( l \) on the patty paper. Label a point not on the line \( A \).
2. Fold the patty paper so the line folds onto itself through point \( A \). Be sure to crease the paper.
3. Unfold the patty paper.
4. Fold the new line onto itself through point \( A \).
5. Unfold the patty paper.
6. Use your straightedge to mark the second creased line.
7. Label the new line \( m \).

Line \( m \) is parallel to line \( l \).

Common Errors/Misconceptions

- inappropriately changing the compass setting
- moving the patty paper before completing the construction
- not creating large enough arcs to find the point of intersection
- attempting to measure lengths and angles with rulers and protractors
Guided Practice 1.2.3

Example 1

Use a compass and a straightedge to construct the perpendicular bisector of $\overline{AB}$.

1. Make a large arc intersecting $\overline{AB}$.
   
   Put the sharp point of your compass on endpoint $A$. Open the compass wider than half the distance of $\overline{AB}$. Draw the arc.
2. Make a second large arc.

Without changing your compass setting, put the sharp point of the compass on endpoint $B$.

Make a second large arc.

It is important that the arcs intersect each other in two places.
3. Connect the points of intersection of the arcs.
   
   Use your straightedge to connect the points of intersection of the arcs.
   
   Label the new line $m$.

   Do not erase any of your markings.

   Line $m$ is the perpendicular bisector of $AB$. 

Example 2

Use a compass and a straightedge to construct a line perpendicular to line $\ell$ through point $A$.

1. Draw line $\ell$ with point $A$ on the line.

2. Make an arc on either side of point $A$.
   - Put the sharp point of your compass on point $A$.
   - Make arcs on either side of point $A$ through line $\ell$.
   - Label the points of intersection $C$ and $D$.

3. Make a set of arcs on either side of line $\ell$.
   - Place the sharp point of the compass on point $C$.
   - Open the compass so it extends beyond point $A$.
   - Create an arc on either side of the line.
4. Make a second set of arcs on either side of line \( l \).

Without changing your compass setting, put the sharp point of the compass on point \( D \).

Make an arc on either side of the line.

It is important that the arcs intersect each other.

5. Connect the points of intersection.

Use your straightedge to connect the points of intersection of the arcs.

This line should also go through point \( A \).

Label the new line \( m \).

Do not erase any of your markings.

Line \( l \) is perpendicular to line \( m \) through point \( A \).
Example 3
Use a compass and a straightedge to construct a line perpendicular to line \( \ell \) through point \( B \) that is not on the line.

1. Draw line \( \ell \) with point \( B \) not on the line.

2. Make a large arc that intersects line \( \ell \).
   - Put the sharp point of your compass on point \( B \).
   - Open the compass until it extends farther than line \( \ell \).
   - Make a large arc that intersects the given line in exactly two places.
   - Label the points of intersection \( F \) and \( G \).
3. Make a set of arcs above line $m$.

Without changing your compass setting, put the sharp point of the compass on point $F$. Make a second arc above the given line.

Without changing your compass setting, put the sharp point of the compass on point $G$. Make a third arc above the given line. The third arc must intersect the second arc.

Label the point of intersection $H$.

4. Draw the perpendicular line.

Use your straightedge to connect points $B$ and $H$.

Label the new line $\ell$.

Do not erase any of your markings.

Line $\ell$ is perpendicular to line $m$. 
Example 4

Use a compass and a straightedge to construct a line parallel to line $\ell$ through point $C$ that is not on the line.

1. Draw line $\ell$ with point $C$ not on the line.

2. Construct a line perpendicular to line $\ell$ through point $C$.
   Make a large arc that intersects line $\ell$.
   Put the sharp point of your compass on point $C$.
   Open the compass until it extends farther than line $\ell$.
   Make a large arc that intersects the given line in exactly two places.
   Label the points of intersection $J$ and $K$.

3. Make a set of arcs below line $\ell$.
   Without changing your compass setting, put the sharp point of the compass on point $J$. Make a second arc below the given line.

4. Without changing your compass setting, put the sharp point of the compass on point $K$. Make a third arc below the given line.
   Label the point of intersection $R$. 
3. Draw the perpendicular line.
   Use your straightedge to connect points $C$ and $R$.
   Label the new line $\ell$.

   Do not erase any of your markings.
   Line $n$ is perpendicular to line $\ell$.

4. Construct a second line perpendicular to line $\ell$.
   Put the sharp point of your compass on point $C$.
   Make a large arc that intersects line $\ell$ on either side of point $C$.
   Label the points of intersection $X$ and $Y$.

(continued)
Make a set of arcs to the right of line $p$.

Put the sharp point of your compass on point $X$.

Open the compass so that it extends beyond point $C$.

Make an arc to the right of line $p$.

Without changing your compass setting, put the sharp point of the compass on point $Y$. Make another arc to the right of line $p$.

Label the point of intersection $S$. 
5. Draw the perpendicular line.
   Use your straightedge to connect points C and S.
   Label the new line q.

Do not erase any of your markings.
Line q is perpendicular to line p.
Line q is parallel to line \( \alpha \).
Problem-Based Task 1.2.3: Triangle Altitudes

The altitude of a triangle is the perpendicular line from a vertex to its opposite side. The altitude of a triangle is also called the height. What happens when all altitudes of one triangle are constructed?
Problem-Based Task 1.2.3: Triangle Altitudes

Coaching

a. What is a perpendicular line?

b. How many altitudes does a triangle have?

Use the triangle below for the following constructions.

![Triangle ABC]

C

A

B

c. What is the process for constructing the altitude from point B to $\overline{AC}$?

d. Construct the altitude from point B to $\overline{AC}$.

e. Construct the altitude from point C to $\overline{AB}$.

f. Construct the altitude from point A to $\overline{BC}$.

g. What happens when all altitudes of the triangle are constructed?

h. Is this true for all triangles?

i. How can you prove your answer to be correct?
Problem-Based Task 1.2.3: Triangle Altitudes

Coaching Sample Responses

a. What is a perpendicular line?
   A perpendicular line is the shortest distance from a point to a line.
   Perpendicular lines create a 90° angle.

b. How many altitudes does a triangle have?
   A triangle has 3 altitudes.
   Each altitude extends from each vertex to its opposite side.

Use the triangle below for the following constructions.

[Diagram of a triangle ABC]

C. What is the process for constructing the altitude from point B to \(\overline{AC}\)?
   The construction of the altitude is the same as constructing a perpendicular line from point B to \(\overline{AC}\).
   Put the sharp point of the compass on point B.
   Open the compass until it extends farther than \(\overline{AC}\).
   Make a large arc that intersects \(\overline{AC}\) in exactly two places. If the arc does not intersect \(\overline{AC}\),
   extend \(\overline{AC}\).
   Without changing your compass setting, put the sharp point of the compass on one point of intersection.
   Make a second arc below the given line.
   Without changing your compass setting, put the sharp point of the compass on the second point of intersection.
   Make a third arc below the given line.
   Use a straightedge to connect the arc below \(\overline{AC}\) and point B.
   Do not erase any of your markings.
d. Construct the altitude from point \( B \) to \( \overline{AC} \).
   Check students’ papers for accuracy.

e. Construct the altitude from point \( C \) to \( \overline{AB} \).
   Check students’ papers for accuracy.

f. Construct the altitude from point \( A \) to \( \overline{BC} \).
   Check students’ papers for accuracy.

g. What happens when all altitudes of the triangle are constructed?
   If constructed correctly, all three altitudes will intersect at the same point.

h. Is this true for all triangles?
   The three altitudes of every triangle will always intersect at one point.
   Sometimes the altitudes will intersect outside of the triangle.

i. How can you prove your answer to be correct?
   Draw another triangle and construct each altitude.
   Compare results with others.

**Recommended Closure Activity**

Select one or more of the essential questions for a class discussion or as a journal entry prompt.
Practice 1.2.3: Constructing Perpendicular and Parallel Lines

Use a compass and a straightedge to copy each segment, and then construct the perpendicular bisector of each.

1. \(\overline{AB}\)

2. \(\overline{CD}\)

Use a compass and a straightedge to copy each segment, place a point on the segment, and then construct a perpendicular line through the point.

3. \(\overline{EF}\)

4. \(\overline{GH}\)

Use a compass and a straightedge to copy each segment, place a point not on the segment, and then construct a perpendicular line through the point.

5. \(\overline{JK}\)

6. \(\overline{MN}\)

Use a compass and a straightedge to copy each segment, place a point not on the segment, and then construct a parallel line through the point.

7. \(\overline{PQ}\)

8. \(\overline{RS}\)

9. \(\overline{TU}\)

10. \(\overline{VW}\)
UNIT 1 • SIMILARITY, CONGRUENCE, AND PROOFS

Lesson 3: Constructing Polygons

Common Core Georgia Performance Standard
MCC9–12.G.CO.13

Essential Questions
1. How can you justify that a construction was done correctly?
2. How can a polygon be constructed given a circle?
3. How are basic constructions used to construct regular polygons?

WORDS TO KNOW

circle the set of all points in a plane that are equidistant from a reference point in that plane, called the center. The set of points forms a two-dimensional curve that measures 360°.

congruent having the same shape, size, or angle

construction a precise representation of a figure using a straightedge and compass, patty paper and a straightedge, or a reflecting device and a straightedge

diameter a straight line passing through the center of a circle connecting two points on the circle; equal to twice the radius

equilateral triangle a triangle with all three sides equal in length

inscribe to draw one figure within another figure so that every vertex of the enclosed figure touches the outside figure

radius a line segment that extends from the center of a circle to a point on the circle. Its length is half the diameter.

regular hexagon a six-sided polygon with all sides equal and all angles measuring 120°

regular polygon a two-dimensional figure with all sides and all angles congruent

square a four-sided regular polygon with all sides equal and all angles measuring 90°

triangle a three-sided polygon with three angles
Recommended Resources

- DePaul University. “Inscribing Regular Polygons.”
  
  
  This site includes overviews and graphics demonstrating how to construct various regular polygons in a given circle.

- Math Open Reference. “Hexagon inscribed in a circle.”
  
  
  This site provides animated step-by-step instructions for constructing a regular hexagon inscribed in a given circle.

- Zef Damen. “Equilateral Triangle.”
  
  
  This site gives step-by-step instructions for constructing an equilateral triangle inscribed in a given circle.
Lesson 1.3.1: Constructing Equilateral Triangles Inscribed in Circles

Warm-Up 1.3.1

The town of Fairside is planning an outdoor concert in the park. During the planning process, committee members determined there will be two large speakers, but they can’t decide where the audience should sit. The best arrangement for two speakers and the center of the audience is a triangle where each angle is 60°. The diagram below depicts the line segment formed by the two speakers; a 60° angle is also shown.

1. Use the 60° angle and the given segment to construct the triangle created by the two speakers and the center of the audience.

2. Triangles are said to be congruent if the angle measures of both triangles are the same and the lengths of the sides are the same. Is it possible to construct a second non-congruent triangle using the given information? Explain your reasoning.
Lesson 1.3.1: Constructing Equilateral Triangles Inscribed in Circles

Common Core Georgia Performance Standard

MCC9–12.G.CO.13

Warm-Up 1.3.1 Debrief

The town of Fairside is planning an outdoor concert in the park. During the planning process, committee members determined there will be two large speakers, but they can’t decide where the audience should sit. The best arrangement for two speakers and the center of the audience is a triangle where each angle is 60°. The diagram below depicts the line segment formed by the two speakers; a 60° angle is also shown.

1. Use the 60° angle and the given segment to construct the triangle created by the two speakers and the center of the audience.

   The speakers represent two of the three vertices of the triangle.

   Use the segment that joins each of the speakers as one side of the triangle.

   Copy the given 60° angle using construction methods previously learned.

   Use the point on Speaker 1 as the vertex of one of the angles.

   Copy the given 60° angle a second time, using the point on Speaker 2 as the vertex of the second angle.

   Extend the sides of the copied angles to find the point of intersection.

   The point of intersection represents the center of the audience.
2. Triangles are said to be congruent if the angle measures of both triangles are the same and the lengths of the sides are the same. Is it possible to construct a second non-congruent triangle using the given information? Explain your reasoning.

It is not possible to construct a second non-congruent triangle.

The given information includes two angle measures and a side length between those two angles. It is not possible to construct a different triangle with these measurements.

If you have two triangles and any two angles and the included side are equal, then the triangles are congruent. This concept will be explained in more detail later in the unit.

**Connection to the Lesson**

- Students will practice geometric constructions by copying angles.
- Students will also learn more about equilateral triangles in this lesson.
Prerequisite Skills
This lesson requires the use of the following skills:

- using a compass
- copying and bisecting line segments
- constructing perpendicular lines
- constructing circles of a given radius

Introduction
The ability to copy and bisect angles and segments, as well as construct perpendicular and parallel lines, allows you to construct a variety of geometric figures, including triangles, squares, and hexagons. There are many ways to construct these figures and others. Sometimes the best way to learn how to construct a figure is to try on your own. You will likely discover different ways to construct the same figure and a way that is easiest for you. In this lesson, you will learn two methods for constructing a triangle within a circle.

Key Concepts

Triangles

- A **triangle** is a polygon with three sides and three angles.
- There are many types of triangles that can be constructed.
- Triangles are classified based on their angle measure and the measure of their sides.
- **Equilateral triangles** are triangles with all three sides equal in length.
- The measure of each angle of an equilateral triangle is 60°.

Circles

- A circle is the set of all points that are equidistant from a reference point, the center.
- The set of points forms a two-dimensional curve that is 360°.
- Circles are named by their center. For example, if a circle has a center point, G, the circle is named circle G.
- The **diameter** of a circle is a straight line that goes through the center of a circle and connects two points on the circle. It is twice the radius.
The radius of a circle is a line segment that runs from the center of a circle to a point on the circle.

The radius of a circle is one-half the length of the diameter.

There are 360° in every circle.

**Inscribing Figures**

- To **inscribe** means to draw a figure within another figure so that every vertex of the enclosed figure touches the outside figure.

- A figure inscribed within a circle is a figure drawn within a circle so that every vertex of the figure touches the circle.

- It is possible to inscribe a triangle within a circle. Like with all constructions, the only tools used to inscribe a figure are a straightedge and a compass, patty paper and a straightedge, reflective tools and a straightedge, or technology.

- This lesson will focus on constructions with a compass and a straightedge.

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**Method 1: Constructing an Equilateral Triangle Inscribed in a Circle Using a Compass**

1. To construct an equilateral triangle inscribed in a circle, first mark the location of the center point of the circle. Label the point \( X \).
2. Construct a circle with the sharp point of the compass on the center point.
3. Label a point on the circle point \( A \).
4. Without changing the compass setting, put the sharp point of the compass on \( A \) and draw an arc to intersect the circle at two points. Label the points \( B \) and \( C \).
5. Use a straightedge to construct \( \overline{BC} \).
6. Put the sharp point of the compass on point \( B \). Open the compass until it extends to the length of \( \overline{BC} \). Draw another arc that intersects the circle at two points. Label the point \( D \).
7. Use a straightedge to construct \( \overline{BD} \) and \( \overline{CD} \).

Do not erase any of your markings.

Triangle \( BCD \) is an equilateral triangle inscribed in circle \( X \).
• A second method “steps out” each of the vertices.
• Once a circle is constructed, it is possible to divide the circle into 6 equal parts.
• Do this by choosing a starting point on the circle and moving the compass around the circle, making marks that are the length of the radius apart from one another.
• Connecting every other point of intersection results in an equilateral triangle.

Method 2: Constructing an Equilateral Triangle Inscribed in a Circle Using a Compass

1. To construct an equilateral triangle inscribed in a circle, first mark the location of the center point of the circle. Label the point \( X \).
2. Construct a circle with the sharp point of the compass on the center point.
3. Label a point on the circle point \( A \).
4. Without changing the compass setting, put the sharp point of the compass on \( A \) and draw an arc to intersect the circle at one point. Label the point of intersection \( B \).
5. Put the sharp point of the compass on point \( B \) and draw an arc to intersect the circle at one point. Label the point of intersection \( C \).
6. Continue around the circle, labeling points \( D \), \( E \), and \( F \). Be sure not to change the compass setting.
7. Use a straightedge to connect \( A \) and \( C \), \( C \) and \( E \), and \( E \) and \( A \). Do not erase any of your markings.

Triangle \( ACE \) is an equilateral triangle inscribed in circle \( X \).

Common Errors/Misconceptions
• inappropriately changing the compass setting
• attempting to measure lengths and angles with rulers and protractors
• not creating large enough arcs to find the points of intersection
Guided Practice 1.3.1

Example 1

Construct equilateral triangle $ACE$ inscribed in circle $O$ using Method 1.

1. Construct circle $O$.
   Mark the location of the center point of the circle, and label the point $O$.
   Construct a circle with the sharp point of the compass on the center point.

2. Label a point on the circle point $Z$. 

---

Guided Practice 1.3.1
3. Locate vertices $A$ and $C$ of the equilateral triangle.

Without changing the compass setting, put the sharp point of the compass on $Z$ and draw an arc to intersect the circle at two points. Label the points $A$ and $C$.

4. Locate the third vertex of the equilateral triangle.

Put the sharp point of the compass on point $A$. Open the compass until it extends to the length of $AC$. Draw another arc that intersects the circle, and label the point $E$.

5. Construct the sides of the triangle.

Use a straightedge to connect $A$ and $C$, $C$ and $E$, and $A$ and $E$. Do not erase any of your markings.

Triangle $ACE$ is an equilateral triangle inscribed in circle $O$. 

\[ \checkmark \]
Example 2

Construct equilateral triangle $ACE$ inscribed in circle $O$ using Method 2.

1. Construct circle $O$.

   Mark the location of the center point of the circle, and label the point $O$. Construct a circle with the sharp point of the compass on the center point.

2. Label a point on the circle point $A$. 
3. Locate vertices $C$ and $E$.

Begin by marking the remaining five equidistant points around the circle. Without changing the compass setting, put the sharp point of the compass on $A$. Draw an arc to intersect the circle at one point. Label the point of intersection $B$.

Put the sharp point of the compass on point $B$. Without changing the compass setting, draw an arc to intersect the circle at one point. Label the point of intersection $C$.

Continue around the circle, labeling points $D$, $E$, and $F$. Be sure not to change the compass setting.
4. Construct the sides of the triangle.

Use a straightedge to connect $A$ and $C$, $C$ and $E$, and $E$ and $A$. Do not erase any of your markings.

Triangle $ACE$ is an equilateral triangle inscribed in circle $O$. 

[Diagram of a triangle inscribed in a circle with labels $A$, $B$, $C$, $D$, and $E$]
Example 3

Construct equilateral triangle $JKL$ inscribed in circle $P$ using Method 1. Use the length of $HP$ as the radius for circle $P$.

1. Construct circle $P$.

Mark the location of the center point of the circle, and label the point $P$. Set the opening of the compass equal to the length of $HP$. Then, put the sharp point of the compass on point $P$ and construct a circle. Label a point on the circle point $G$. 
2. Locate vertices $J$ and $K$ of the equilateral triangle.

Without changing the compass setting, put the sharp point of the compass on $G$. Draw an arc to intersect the circle at two points. Label the points $J$ and $K$.

3. Locate the third vertex of the equilateral triangle.

Put the sharp point of the compass on point $J$. Open the compass until it extends to the length of $JK$. Draw another arc that intersects the circle, and label the point $L$. 
4. Construct the sides of the triangle.

Use a straightedge to connect $J$ and $K$, $K$ and $L$, and $L$ and $J$. Do not erase any of your markings.

Triangle $JKL$ is an equilateral triangle inscribed in circle $P$ with the given radius.
Example 4

Construct equilateral triangle $JLN$ inscribed in circle $P$ using Method 2. Use the length of $\overline{HP}$ as the radius for circle $P$.

1. Construct circle $P$.
   Mark the location of the center point of the circle, and label the point $P$. Set the opening of the compass equal to the length of $\overline{HP}$. Then, put the sharp point of the compass on point $P$ and construct a circle. Label a point on the circle point $G$. 
2. Locate vertex $J$.

Without changing the compass setting, put the sharp point of the compass on $G$. Draw an arc to intersect the circle at one point. Label the point of intersection $J$.

Put the sharp point of the compass on point $J$. Without changing the compass setting, draw an arc to intersect the circle at one point. Label the point of intersection $K$.

Continue around the circle, labeling points $L$, $M$, and $N$. Be sure not to change the compass setting.
3. Construct the sides of the triangle.

Use a straightedge to connect \( J \) and \( L \), \( L \) and \( N \), and \( J \) and \( N \). Do not erase any of your markings.

Triangle \( JLN \) is an equilateral triangle inscribed in circle \( P \).
Problem-Based Task 1.3.1: Vending Machine Placement

As an employee of a skating rink, Jarno was asked to determine the placement of 3 vending machines. Each of the 3 machines needs to be placed along the edge of the circular skating rink. The distance between each machine must be the same. Where should Jarno place each machine? A diagram of the skating rink is provided below.
Problem-Based Task 1.3.1: Vending Machine Placement

Coaching

a. Suppose the 3 vending machines were placed around the edge of the circular skating rink. If a customer were to skate from the first vending machine to the second, to the third, and back to the first machine, what figure would be created?

b. If the distance between each of the machines is the same, what is true about the length of each of the sides of the figure the customer skated?

c. Which geometric construction can be carried out to identify the locations of the 3 vending machines?

d. What is the procedure for making this geometric construction?

e. Where should Jarno place each machine?
Problem-Based Task 1.3.1: Vending Machine Placement

Coaching Sample Responses

a. Suppose the 3 vending machines were placed around the edge of the circular skating rink. If a customer were to skate from the first vending machine to the second, to the third, and back to the first machine, what figure would be created?

If a customer were to skate from the first vending machine to the second, to the third, and back to the first machine, they would create a triangle.

b. If the distance between each of the machines is the same, what is true about the length of each of the sides of the figure the customer skated?

Each side of the triangle has the same length.

c. Which geometric construction can be carried out to identify the locations of the 3 vending machines?

Constructing an equilateral triangle inscribed in a circle would identify the locations of the 3 vending machines.

d. What is the procedure for making this geometric construction?

There are two ways of constructing an equilateral triangle inscribed in a circle. Either is correct.

**Method 1**

- Label a point on the circle point $A$.
- Put the sharp point of the compass on $A$ and set the compass width equal to the radius of the circle.
- Keeping the sharp point on $A$, draw an arc to intersect the circle at two points. Label the points $B$ and $C$.
- Use a straightedge to construct $\overline{BC}$.
- Put the sharp point of the compass on point $B$. Open the compass until it extends to the length of $\overline{BC}$. Draw another arc that intersects the circle. Label the point $D$.
- Use a straightedge to construct $\overline{BD}$ and $\overline{CD}$.
- Triangle $BCD$ is an equilateral triangle constructed within the circle. See the illustration on the following page.
Method 2

- Label a point on the circle point $A$.
- Put the sharp point of the compass on $A$ and set the compass width equal to the radius of the circle.
- Keeping the sharp point on $A$, draw an arc to intersect the circle at one point. Label the point of intersection $B$.
- Put the sharp point of the compass on point $B$. Without changing the compass setting, draw an arc to intersect the circle at one point. Label the point of intersection $C$.
- Continue around the circle, labeling points $D$, $E$, and $F$. Be sure not to change the compass setting.
- Use a straightedge to connect $B$ and $D$, $D$ and $F$, and $F$ and $B$.
- Triangle $BDF$ is an equilateral triangle constructed within the circle. See the illustration on the following page.
e. Where should Jarno place each machine?

Jarno should place the vending machines so that the machines create an equilateral triangle. Based on the diagrams, the vending machines should be placed at points $B, C,$ and $D$ for the first method, and at points $B, D,$ and $F$ for the second method.

**Recommended Closure Activity**

Select one or more of the essential questions for a class discussion or as a journal entry prompt.
Practice 1.3.1: Constructing Equilateral Triangles Inscribed in Circles
Use a compass and a straightedge to construct each equilateral triangle using Method 1.

1. Construct equilateral triangle $BCD$ inscribed in circle $Z$.

2. Construct equilateral triangle $FGH$ inscribed in circle $Y$ with radius $\overline{AB}$.

3. Construct equilateral triangle $JKL$ inscribed in circle $X$ with radius $\overline{CD}$.

4. Construct equilateral triangle $NOP$ inscribed in circle $W$ with the radius equal to twice $\overline{EF}$.

5. Construct equilateral triangle $RST$ inscribed in circle $V$ with the radius equal to one-half $\overline{GH}$.

continued
Use a compass and a straightedge to construct each equilateral triangle using Method 2.

6. Construct equilateral triangle $BCD$ inscribed in circle $Z$.

7. Construct equilateral triangle $FGH$ inscribed in circle $Y$ with radius $\overline{AB}$.

8. Construct equilateral triangle $JKL$ inscribed in circle $X$ with radius $\overline{CD}$.

9. Construct equilateral triangle $NOP$ inscribed in circle $W$ with the radius equal to twice $\overline{EF}$.

10. Construct equilateral triangle $RST$ inscribed in circle $V$ with the radius equal to one-half $\overline{GH}$.
Antonia is making four corner tables, one for each of her three sisters and herself. She has one large square piece of wood that she plans to cut into four tabletops. She begins by marking the needed cuts for the tabletops on the square piece of wood.

1. Each angle of the square piece of wood measures $90^\circ$. If Antonia bisects one angle of the square, what is the measure of the two new angles?

2. Bisect one angle of the square. Extend the angle bisector so that it intersects the square in two places. Where does the bisector intersect the square?

3. Bisect the remaining angles of the square. If Antonia cuts along each angle bisector, what figures will she have created?

4. What are the measures of each of the angles of the new figures?
Lesson 1.3.2: Constructing Squares Inscribed in Circles

Common Core Georgia Performance Standard

MCC9–12.G.CO.13

Warm-Up 1.3.2 Debrief

Antonia is making four corner tables, one for each of her three sisters and herself. She has one large square piece of wood that she plans to cut into four tabletops. She begins by marking the needed cuts for the tabletops on the square piece of wood.

1. Each angle of the square piece of wood measures 90°. If Antonia bisects one angle of the square, what is the measure of the two new angles?

   When an angle is bisected, it is divided into two equal angles.

   \[90 \div 2 = 45\]

   Each of the new angles measures 45°.

2. Bisect one angle of the square. Extend the angle bisector so that it intersects the square in two places. Where does the bisector intersect the square?

   The angle bisector intersects the square at the bisected angle and the angle opposite the bisected angle.
3. Bisect the remaining angles of the square. If Antonia cuts along each angle bisector, what figures will she have created?

   Antonia will have created 4 triangles.

4. What are the measures of each of the angles of the new figures?

   Each triangle will have two 45° angles and one 90° angle.

**Connection to the Lesson**

- Students will practice geometric constructions by bisecting angles.
- Students will also learn more about squares in this lesson.
Introduction

Triangles are not the only figures that can be inscribed in a circle. It is also possible to inscribe other figures, such as squares. The process for inscribing a square in a circle uses previously learned skills, including constructing perpendicular bisectors.

Key Concepts

- A **square** is a four-sided regular polygon.
- A regular polygon is a polygon that has all sides equal and all angles equal.
- The measure of each of the angles of a square is 90°.
- Sides that meet at one angle to create a 90° angle are perpendicular.
- By constructing the perpendicular bisector of a diameter of a circle, you can construct a square inscribed in a circle.
### Constructing a Square Inscribed in a Circle Using a Compass

<table>
<thead>
<tr>
<th>Step</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.</td>
<td>To construct a square inscribed in a circle, first mark the location of the center point of the circle. Label the point ( X ).</td>
</tr>
<tr>
<td>2.</td>
<td>Construct a circle with the sharp point of the compass on the center point.</td>
</tr>
<tr>
<td>3.</td>
<td>Label a point on the circle point ( A ).</td>
</tr>
<tr>
<td>4.</td>
<td>Use a straightedge to connect point ( A ) and point ( X ). Extend the line through the circle, creating the diameter of the circle. Label the second point of intersection ( C ).</td>
</tr>
<tr>
<td>5.</td>
<td>Construct the perpendicular bisector of ( AC ) by putting the sharp point of your compass on endpoint ( A ). Open the compass wider than half the distance of ( AC ). Make a large arc intersecting ( AC ). Without changing your compass setting, put the sharp point of the compass on endpoint ( C ). Make a second large arc. Use your straightedge to connect the points of intersection of the arcs.</td>
</tr>
<tr>
<td>6.</td>
<td>Extend the bisector so it intersects the circle in two places. Label the points of intersection ( B ) and ( D ).</td>
</tr>
<tr>
<td>7.</td>
<td>Use a straightedge to connect points ( A ) and ( B ), ( B ) and ( C ), ( C ) and ( D ), and ( A ) and ( D ). Do not erase any of your markings. Quadrilateral ( ABCD ) is a square inscribed in circle ( X ).</td>
</tr>
</tbody>
</table>

### Common Errors/Misconceptions
- inappropriately changing the compass setting
- attempting to measure lengths and angles with rulers and protractors
- not creating large enough arcs to find the points of intersection
- not extending segments long enough to find the vertices of the square
Guided Practice 1.3.2

Example 1

Construct square $ABCD$ inscribed in circle $O$.

1. Construct circle $O$.
   
   Mark the location of the center point of the circle, and label the point $O$. Construct a circle with the sharp point of the compass on the center point.

2. Label a point on the circle point $A$. 
3. Construct the diameter of the circle.
   Use a straightedge to connect point $A$ and point $O$. Extend the line through the circle, creating the diameter of the circle. Label the second point of intersection $C$.

4. Construct the perpendicular bisector of $AC$.
   Extend the bisector so it intersects the circle in two places. Label the points of intersection $B$ and $D$. 
5. Construct the sides of the square.

Use a straightedge to connect points $A$ and $B$, $B$ and $C$, $C$ and $D$, and $A$ and $D$. Do not erase any of your markings.

Quadrilateral $ABCD$ is a square inscribed in circle $O$. 

$\checkmark$
Example 2

Construct square $EFGH$ inscribed in circle $P$ with the radius equal to the length of $EP$.

1. Construct circle $P$.

   Mark the location of the center point of the circle, and label the point $P$. Set the opening of the compass equal to the length of $EP$. Construct a circle with the sharp point of the compass on the center point, $P$.

2. Label a point on the circle point $E$. 

   $E$
3. Construct the diameter of the circle.
   Use a straightedge to connect point $E$ and point $P$. Extend the line through the circle, creating the diameter of the circle. Label the second point of intersection $G$.

![Diagram of a circle with diameter connecting $E$ and $P$, and point $G$ labeled on the extension of the line]

4. Construct the perpendicular bisector of $EG$.
   Extend the bisector so it intersects the circle in two places. Label the points of intersection $F$ and $H$.

![Diagram of a circle with perpendicular bisector $FG$ and points $F$, $G$, $H$ labeled on the circle]
5. Construct the sides of the square.

Use a straightedge to connect points $E$ and $F$, $F$ and $G$, $G$ and $H$, and $H$ and $E$. Do not erase any of your markings.

Quadrilateral $EFGH$ is a square inscribed in circle $P$. 

☑
**Example 3**

Construct square $JKLM$ inscribed in circle $Q$ with the radius equal to one-half the length of $\overline{JL}$.

1. Construct circle $Q$.
   
   Mark the location of the center point of the circle, and label the point $Q$. Bisect the length of $\overline{JL}$. Label the midpoint of the segment as point $P$.

   Next, set the opening of the compass equal to the length of $\overline{JP}$. Construct a circle with the sharp point of the compass on the center point, $Q$. 

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**Instruction**
2. Label a point on the circle point $J$.

3. Construct the diameter of the circle.
   Use a straightedge to connect point $J$ and point $Q$. Extend the line through the circle, creating the diameter of the circle. Label the second point of intersection $L$. 
4. Construct the perpendicular bisector of $\overline{JL}$.
   Extend the bisector so it intersects the circle in two places. Label the points of intersection $K$ and $M$.

5. Construct the sides of the square.

Quadrilateral $JKLM$ is a square inscribed in circle $Q$. 
Problem-Based Task 1.3.2: Constructing a Regular Octagon

A regular octagon is a polygon with eight sides that are equal in length and eight angles that are equal in measure. How could you construct the largest octagon possible within the given square?
Problem-Based Task 1.3.2: Constructing a Regular Octagon

Coaching

a. How could you describe the point at which each angle bisector of the square intersects? Locate this point.

b. Which basic construction method could be used to locate the midpoint of each side of the square? Locate the midpoint of each side.

c. If a segment were drawn from each midpoint to the midpoint on the opposite side, where would the segments intersect? Locate this point.

d. How could you determine the distance from the center of the square to the midpoint of one of the sides of the square?

e. How could this distance help you determine where each remaining angle of the regular octagon is located? Locate the remaining angles of the octagon.
Problem-Based Task 1.3.2: Constructing a Regular Octagon

Coaching Sample Responses

a. How could you describe the point at which each angle bisector of the square intersects? Locate this point.

The point at which the angle bisectors of the square intersect can be described as the center of the square.

b. Which basic construction method could be used to locate the midpoint of each side of the square? Locate the midpoint of each side.

Construct the midpoint of each side by finding the perpendicular bisector of each side.
c. If a segment were drawn from each midpoint to the midpoint on the opposite side, where would the segments intersect? Locate this point.

The segments would intersect at the center of the square.

d. How could you determine the distance from the center of the square to the midpoint of one of the sides of the square?

Put the sharp point of the compass at the center of the square.

Open the compass wide enough to touch the midpoint of one side of the square.
e. How could this distance help you determine where each remaining angle of the regular octagon is located? Locate the remaining angles of the octagon.

Keep the sharp point of the compass at the center of the square. Draw an arc that intersects each of the angle bisectors previously drawn.

The points of intersection create the remaining vertices of the regular octagon.

Connect the vertices to construct the regular octagon.

**Recommended Closure Activity**

Select one or more of the essential questions for a class discussion or as a journal entry prompt.
Practice 1.3.2: Constructing Squares Inscribed in Circles

Use a compass and a straightedge to construct each square inscribed in a circle.

1. Construct square $ABCD$ inscribed in circle $E$.

2. Construct square $FGHJ$ inscribed in circle $K$.

3. Construct square $LMNO$ inscribed in circle $P$ with radius $LP$.

4. Construct square $QRST$ inscribed in circle $U$ with radius $QU$.

5. Construct square $VWXYZ$ inscribed in circle $Z$ with radius $VZ$.

6. Construct square $BCDE$ inscribed in circle $F$ with radius $BF$.

7. Construct square $GHJK$ inscribed in circle $L$ with the radius equal to twice $AB$.

8. Construct square $MNOP$ inscribed in circle $Q$ with the radius equal to one-half $CD$.

9. Construct square $RSTU$ inscribed in circle $V$ with the diameter equal to $RT$.

10. Construct square $WXYZ$ inscribed in circle $A$ with the diameter equal to $WY$. 
Lesson 1.3.3: Constructing Regular Hexagons Inscribed in Circles

Warm-Up 1.3.3

The developers of a community garden would like to construct planting beds using only donated materials. One planting bed will be square. Developers would like to create the square planting bed using only one plank of wood, represented by the line below. Use the given segment to construct a design for the square planting bed.

1. Use a compass and a straightedge to determine the length of each side of the planting bed.

2. Describe the process for determining the length of each side of the planting bed.
Lesson 1.3.3: Constructing Regular Hexagons Inscribed in Circles

Common Core Georgia Performance Standard

MCC9–12.G.CO.13

Warm-Up 1.3.3 Debrief

The developers of a community garden would like to construct planting beds using only donated materials. One planting bed will be square. Developers would like to create the square planting bed using only one plank of wood, represented by the segment below. Use the given segment to construct a design for the square planting bed.

1. Use a compass and a straightedge to determine the length of each side of the planting bed.
   Bisect the original segment, and then bisect each half.
   The length of each side of the square planting bed will be \( \frac{1}{4} \) the length of the original plank of wood.
2. Describe the process for determining the length of each side of the planting bed.

To determine the length of each side of the square planting bed, first find the midpoint of the plank of wood by bisecting the plank. This construction will divide the plank into two equal pieces.
Next, find the midpoint of one of the halves by bisecting it.

The result is a piece of the plank that is \( \frac{1}{4} \) the original length of the plank of wood.

**Connection to the Lesson**

- Students will practice geometric constructions and apply these methods in the construction of regular hexagons inscribed in a circle.
Introduction

Construction methods can also be used to construct figures in a circle. One figure that can be inscribed in a circle is a hexagon. Hexagons are polygons with six sides.

Key Concepts

- **Regular hexagons** have six equal sides and six angles, each measuring 120°.
- The process for inscribing a regular hexagon in a circle is similar to that of inscribing equilateral triangles and squares in a circle.
- The construction of a regular hexagon is the result of the construction of two equilateral triangles inscribed in a circle.

**Method 1: Constructing a Regular Hexagon Inscribed in a Circle Using a Compass**

1. To construct a regular hexagon inscribed in a circle, first mark the location of the center point of the circle. Label the point \(X\).
2. Construct a circle with the sharp point of the compass on the center point.
3. Label a point on the circle point \(A\).
4. Use a straightedge to connect point \(A\) and point \(X\). Extend the line through the circle, creating the diameter of the circle. Label the second point of intersection \(D\).
5. Without changing the compass setting, put the sharp point of the compass on \(A\). Draw an arc to intersect the circle at two points. Label the points \(B\) and \(F\).
6. Put the sharp point of the compass on \(D\). Without changing the compass setting, draw an arc to intersect the circle at two points. Label the points \(C\) and \(E\).
7. Use a straightedge to connect points \(A\) and \(B\), \(B\) and \(C\), \(C\) and \(D\), \(D\) and \(E\), \(E\) and \(F\), and \(F\) and \(A\).

Do not erase any of your markings.

Hexagon \(ABCDEF\) is regular and is inscribed in circle \(X\).
• A second method “steps out” each of the vertices.
• Once a circle is constructed, it is possible to divide the circle into six equal parts.
• Do this by choosing a starting point on the circle and moving the compass around the circle, making marks equal to the length of the radius.
• Connecting every point of intersection results in a regular hexagon.

### Method 2: Constructing a Regular Hexagon Inscribed in a Circle Using a Compass

1. To construct a regular hexagon inscribed in a circle, first mark the location of the center point of the circle. Label the point $X$.
2. Construct a circle with the sharp point of the compass on the center point.
3. Label a point on the circle point $A$.
4. Without changing the compass setting, put the sharp point of the compass on $A$. Draw an arc to intersect the circle at one point. Label the point of intersection $B$.
5. Put the sharp point of the compass on point $B$. Without changing the compass setting, draw an arc to intersect the circle at one point. Label the point of intersection $C$.
6. Continue around the circle, labeling points $D, E,$ and $F$. Be sure not to change the compass setting.
7. Use a straightedge to connect points $A$ and $B$, $B$ and $C$, $C$ and $D$, $D$ and $E$, $E$ and $F$, and $F$ and $A$.

Do not erase any of your markings.

Hexagon $ABCDEF$ is regular and is inscribed in circle $X$.

## Common Errors/Misconceptions

- inappropriately changing the compass setting
- attempting to measure lengths and angles with rulers and protractors
- not creating large enough arcs to find the points of intersection
- not extending segments long enough to find the vertices of the hexagon
Guided Practice 1.3.3

Example 1

Construct regular hexagon $ABCDEF$ inscribed in circle $O$ using Method 1.

1. Construct circle $O$.
   
   Mark the location of the center point of the circle, and label the point $O$. Construct a circle with the sharp point of the compass on the center point.

2. Label a point on the circle point $A$. 
3. Construct the diameter of the circle.
   Use a straightedge to connect point $A$ and the center point, $O$. Extend the line through the circle, creating the diameter of the circle. Label the second point of intersection $D$.

4. Locate two vertices on either side of point $A$.
   Without changing the compass setting, put the sharp point of the compass on point $A$. Draw an arc to intersect the circle at two points. Label the points $B$ and $F$. 
5. Locate two vertices on either side of point $D$.

Without changing the compass setting, put the sharp point of the compass on point $D$. Draw an arc to intersect the circle at two points. Label the points $C$ and $E$.

6. Construct the sides of the hexagon.

Use a straightedge to connect $A$ and $B$, $B$ and $C$, $C$ and $D$, $D$ and $E$, $E$ and $F$, and $F$ and $A$. Do not erase any of your markings.

Hexagon $ABCDEF$ is a regular hexagon inscribed in circle $O$. 
Example 2

Construct regular hexagon $ABCDEF$ inscribed in circle $O$ using Method 2.

1. Construct circle $O$.

Mark the location of the center point of the circle, and label the point $O$. Construct a circle with the sharp point of the compass on the center point.

2. Label a point on the circle point $A$. 
3. Locate the remaining vertices.

Without changing the compass setting, put the sharp point of the compass on $A$. Draw an arc to intersect the circle at one point. Label the point of intersection $B$.

Put the sharp point of the compass on point $B$. Without changing the compass setting, draw an arc to intersect the circle at one point. Label the point of intersection $C$. 

(continued)
Continue around the circle, labeling points $D$, $E$, and $F$. Be sure not to change the compass setting.

4. Construct the sides of the hexagon.

Use a straightedge to connect $A$ and $B$, $B$ and $C$, $C$ and $D$, $D$ and $E$, $E$ and $F$, and $F$ and $A$. Do not erase any of your markings.

Hexagon $ABCDEF$ is a regular hexagon inscribed in circle $O$. 
**Example 3**

Construct regular hexagon $LMNOPQ$ inscribed in circle $R$ using Method 1. Use the length of $RL$ as the radius for circle $R$.

1. Construct circle $R$.

   Mark the location of the center point of the circle, and label the point $R$. Set the opening of the compass equal to the length of $RL$. Put the sharp point of the circle on $R$ and construct a circle.

2. Label a point on the circle point $L$. 

   ![Diagram showing the construction of a regular hexagon inscribed in a circle]

   - $R$ is the center of the circle.
   - $L$ is a point on the circle.
   - $RL$ is the radius of the circle.
3. Construct the diameter of the circle.

Use a straightedge to connect point \( L \) and the center point, \( R \). Extend the line through the circle, creating the diameter of the circle. Label the second point of intersection \( O \).

4. Locate two vertices on either side of point \( L \).

Without changing the compass setting, put the sharp point of the compass on point \( L \). Draw an arc to intersect the circle at two points. Label the points \( M \) and \( Q \).
5. Locate two vertices on either side of point $O$.

Without changing the compass setting, put the sharp point of the compass on point $O$. Draw an arc to intersect the circle at two points. Label the points $P$ and $N$.

6. Construct the sides of the hexagon.

Use a straightedge to connect $L$ and $M$, $M$ and $N$, $N$ and $O$, $O$ and $P$, $P$ and $Q$, and $Q$ and $L$. Do not erase any of your markings.

Hexagon $LMNOPQ$ is a regular hexagon inscribed in circle $R$. 
Example 4

Construct regular hexagon $LMNOPQ$ inscribed in circle $R$ using Method 2. Use the length of $\overline{RL}$ as the radius for circle $R$.

1. Construct circle $R$.
   Mark the location of the center point of the circle, and label the point $R$. Set the opening of the compass equal to the length of $\overline{RL}$. Put the sharp point of the circle on $R$ and construct a circle.

2. Label a point on the circle point $L$. 
3. Locate the remaining vertices.

Without changing the compass setting, put the sharp point of the compass on $L$. Draw an arc to intersect the circle at one point. Label the point of intersection $M$.

Put the sharp point of the compass on point $M$. Without changing the compass setting, draw an arc to intersect the circle at one point. Label the point of intersection $N$. 

(continued)
Continue around the circle, labeling points $O$, $P$, and $Q$. Be sure not to change the compass setting.

4. Construct the sides of the hexagon.

Use a straightedge to connect $L$ and $M$, $M$ and $N$, $N$ and $O$, $O$ and $P$, $P$ and $Q$, and $Q$ and $L$. Do not erase any of your markings.

Hexagon $LMNOPQ$ is a regular hexagon inscribed in circle $R$. 
Problem-Based Task 1.3.3: Constructing a Regular Dodecagon

A regular dodecagon is a polygon with 12 sides that are equal in length and 12 angles that each measure 150°. How could you construct a regular dodecagon?
Problem-Based Task 1.3.3: Constructing a Regular Dodecagon

Coaching

a. How many sides does a regular hexagon have?

b. How does the number of sides of a regular hexagon compare to the number of sides of a regular dodecagon?

c. Suppose a regular hexagon and a regular dodecagon were inscribed in the same circle. How does the length of each side of the regular hexagon compare to the length of each side of the regular dodecagon?

d. What is the process for constructing a regular hexagon?

e. Which basic construction method could you use to construct the 12 sides of the regular dodecagon from the sides of the regular hexagon inscribed in the circle?

f. Use your method to construct a regular dodecagon inscribed in a circle.
Problem-Based Task 1.3.3: Constructing a Regular Dodecagon

Coaching Sample Responses

a. How many sides does a regular hexagon have?
A regular hexagon has six sides.

b. How does the number of sides of a regular hexagon compare to the number of sides of a regular dodecagon?
A regular hexagon has six sides, whereas a regular dodecagon has 12 sides.
The number of sides of a regular hexagon is half the number of sides of a regular dodecagon.

c. Suppose a regular hexagon and a regular dodecagon were inscribed in the same circle. How does the length of each side of the regular hexagon compare to the length of each side of the regular dodecagon?
The length of each side of a regular hexagon is twice the length of each side of a regular dodecagon inscribed in the same circle.

d. What is the process for constructing a regular hexagon?
There are two methods for constructing a regular hexagon. Either method is correct.

Method 1

• Mark the location of the center point of the circle. Label the point X.
• Construct a circle with the sharp point of the compass on the center point.
• Label a point on the circle point A.
• Use a straightedge to connect point A and point X. Extend the line through the circle, creating the diameter of the circle. Label the second point of intersection D.
• Without changing the compass setting, put the sharp point of the compass on A. Draw an arc to intersect the circle at two points. Label the points B and F.
• Put the sharp point of the compass on D. Without changing the compass setting, draw an arc to intersect the circle at two points. Label the points C and E.
• Use a straightedge to connect points A and B, B and C, C and D, D and E, E and F, and F and A.
• Hexagon ABCDEF is a regular hexagon inscribed in circle X.
Method 2

- Mark the location of the center point of the circle. Label the point \( X \).
- Construct a circle with the sharp point of the compass on the center point.
- Label a point on the circle point \( A \).
- Without changing the compass setting, put the sharp point of the compass on \( A \). Draw an arc to intersect the circle at one point. Label the point of intersection \( B \).
- Put the sharp point of the compass on point \( B \). Without changing the compass setting, draw an arc to intersect the circle at one point. Label the point of intersection \( C \).
- Continue around the circle, labeling points \( D \), \( E \), and \( F \). Be sure not to change the compass setting.
- Use a straightedge to connect points \( A \) and \( B \), \( B \) and \( C \), \( C \) and \( D \), \( D \) and \( E \), \( E \) and \( F \), and \( F \) and \( A \).
- Hexagon \( ABCDEF \) is a regular hexagon inscribed in circle \( X \).

e. Which basic construction method could you use to construct the 12 sides of the regular dodecagon from the sides of the regular hexagon inscribed in the circle?

Bisect each of the sides of the regular hexagon inscribed in the circle. Connect the midpoint of each side of the regular hexagon with the center point of the circle. Extend the segments to the circle and mark the intersection points. Connect the intersection points and the vertices of the regular hexagon to construct a regular dodecagon.

Another possible method to construct a dodecagon is to first bisect the length of the radius. Then set the compass to the distance of half the radius and step out 12 points along the circle to represent each of the vertices of the dodecagon.
f. Use your method to construct a regular dodecagon inscribed in a circle.

**Recommended Closure Activity**

Select one or more of the essential questions for a class discussion or as a journal entry prompt.
Practice 1.3.3: Constructing Regular Hexagons Inscribed in Circles

Use a compass and a straightedge to construct each regular hexagon using Method 1.

1. Construct regular hexagon $BCDEFG$ inscribed in circle $Z$.
2. Construct regular hexagon $HJKLMN$ inscribed in circle $Y$ with radius $\overline{AB}$.
3. Construct regular hexagon $PQRSTU$ inscribed in circle $X$ with radius $\overline{CD}$.
4. Construct regular hexagon $DEFGHJ$ inscribed in circle $W$ with the radius equal to twice $\overline{EF}$.
5. Construct regular hexagon $RSTUVW$ inscribed in circle $A$ with the radius equal to one-half $\overline{GH}$.

Use a compass and a straightedge to construct each regular hexagon using Method 2.

6. Construct regular hexagon $BCDEFG$ inscribed in circle $Z$.
7. Construct regular hexagon $HJKLMN$ inscribed in circle $Y$ with radius $\overline{AB}$.
8. Construct regular hexagon $PQRSTU$ inscribed in circle $X$ with radius $\overline{CD}$.
9. Construct regular hexagon $DEFGHJ$ inscribed in circle $W$ with the radius equal to twice $\overline{EF}$.
10. Construct regular hexagon $RSTUVW$ inscribed in circle $A$ with the radius equal to one-half $\overline{GH}$.
UNIT 1 • SIMILARITY, CONGRUENCE, AND PROOFS

Lesson 4: Exploring Congruence

Common Core Georgia Performance Standard

MCC9–12.G.CO.6

Essential Questions

1. What are the differences between rigid and non-rigid motions?
2. How do you identify a transformation as a rigid motion?
3. How do you identify a transformation as a non-rigid motion?

WORDS TO KNOW

angle of rotation the measure of the angle created by the preimage vertex to the point of rotation to the image vertex. All of these angles are congruent when a figure is rotated.

clockwise rotating a figure in the direction that the hands on a clock move

compression a transformation in which a figure becomes smaller; compressions may be horizontal (affecting only horizontal lengths), vertical (affecting only vertical lengths), or both

congruence a transformation in which a geometric figure moves but keeps the same size and shape; a dilation where the scale factor is equal to 1

congruent figures are congruent if they have the same shape, size, lines, and angles; the symbol for representing congruency between figures is ≅

corresponding angles angles of two figures that lie in the same position relative to the figure. In transformations, the corresponding vertices are the preimage and image vertices, so $\angle A$ and $\angle A'$ are corresponding vertices and so on.

corresponding sides sides of two figures that lie in the same position relative to the figure. In transformations, the corresponding sides are the preimage and image sides, so $\overline{AB}$ and $\overline{A'B'}$ are corresponding sides and so on.

clockwise rotating a figure in the opposite direction that the hands on a clock move

dilation a transformation in which a figure is either enlarged or reduced by a scale factor in relation to a center point
UNIT 1 • SIMILARITY, CONGRUENCE, AND PROOFS
Lesson 4: Exploring Congruence

Instruction

- **equidistant** the same distance from a reference point
- **image** the new, resulting figure after a transformation
- **isometry** a transformation in which the preimage and image are congruent
- **line of reflection** the perpendicular bisector of the segments that connect the corresponding vertices of the preimage and the image
- **non-rigid motion** a transformation done to a figure that changes the figure’s shape and/or size
- **point of rotation** the fixed location that an object is turned around; the point can lie on, inside, or outside the figure
- **preimage** the original figure before undergoing a transformation
- **rigid motion** a transformation done to a figure that maintains the figure’s shape and size or its segment lengths and angle measures
- **scale factor** a multiple of the lengths of the sides from one figure to the transformed figure. If the scale factor is larger than 1, then the figure is enlarged. If the scale factor is between 0 and 1, then the figure is reduced.

Recommended Resources

- Math Is Fun. “Rotation.”
  This website explains what a rotation is, and then gives the opportunity to experiment with rotating different shapes about a point of rotation and an angle. There are links at the bottom of the page for translations and reflections with similar applets.

  This site gives a description of a dilation and provides an applet with a slider, allowing users to explore dilating a rectangle with different scale factors. The website goes on to explain how to create a dilation of a polygon.

  This site explains and animates a translation. It also contains links to descriptions and animations of rotations and reflections.
Lesson 1.4.1: Describing Rigid Motions and Predicting the Effects

Warm-Up 1.4.1

Before the digital age, printing presses were used to create text products such as newspapers, brochures, and any other mass-produced, printed material. Printing presses used printing blocks that were the reflection of the image to be printed. Some antique collectors seek out hand-carved printing blocks.

An antique poster of the printed letter “L” was created using a printing block. The “L” has the coordinates $A(2, 5), B(3, 5), C(3, 2), D(5, 2), E(5, 1), \text{ and } F(2, 1)$. Use this information to solve the following problems.

1. What are the coordinates of the printing block through $r_{x-axis}$?

2. Graph the preimage and the image.
Warm-Up 1.4.1 Debrief

Before the digital age, printing presses were used to create text products such as newspapers, brochures, and any other mass-produced, printed material. Printing presses used printing blocks that were the reflection of the image to be printed. Some antique collectors seek out hand-carved printing blocks.

An antique poster of the printed letter “L” was created using a printing block. The “L” has the coordinates \(A(2, 5), B(3, 5), C(3, 2), D(5, 2), E(5, 1),\) and \(F(2, 1)\).

1. What are the coordinates of the printing block through \(r_{x-axis}\)?

\[
\begin{array}{|c|c|}
\hline
r_{x-axis}(x, y) &= (x, -y) \\
\hline
r_{x-axis}[A(2, 5)] &= A'(2, -5) \\
r_{x-axis}[B(3, 5)] &= B'(3, -5) \\
r_{x-axis}[C(3, 2)] &= C'(3, -2) \\
r_{x-axis}[D(5, 2)] &= D'(5, -2) \\
r_{x-axis}[E(5, 1)] &= E'(5, -1) \\
r_{x-axis}[F(2, 1)] &= F'(2, -1) \\
\hline
\end{array}
\]

2. Graph the preimage and the image.

Connection to the Lesson

- Students will be performing reflections in the lesson, but will use lines of reflection other than the \(x\)-axis, \(y\)-axis, and the line \(y = x\).
- Students will use geometric constructions to generate rigid motions rather than using functions.
Introduction

Think about trying to move a drop of water across a flat surface. If you try to push the water droplet, it will smear, stretch, and transfer onto your finger. The water droplet, a liquid, is not rigid. Now think about moving a block of wood across the same flat surface. A block of wood is solid or rigid, meaning it maintains its shape and size when you move it. You can push the block and it will keep the same size and shape as it moves. In this lesson, we will examine rigid motions, which are transformations done to an object that maintain the object’s shape and size or its segment lengths and angle measures.

Key Concepts

- **Rigid motions** are transformations that don’t affect an object’s shape and size. This means that corresponding sides and corresponding angle measures are preserved.
- When angle measures and sides are preserved they are congruent, which means they have the same shape and size.
- The congruency symbol (\(\cong\)) is used to show that two figures are congruent.
- The figure before the transformation is called the preimage.
- The figure after the transformation is the image.
- Corresponding sides are the sides of two figures that lie in the same position relative to the figure. In transformations, the corresponding sides are the preimage and image sides, so \(\overline{AB}\) and \(\overline{A'B'}\) are corresponding sides and so on.
- **Corresponding angles** are the angles of two figures that lie in the same position relative to the figure. In transformations, the corresponding vertices are the preimage and image vertices, so \(\angle A\) and \(\angle A'\) are corresponding vertices and so on.
- Transformations that are rigid motions are translations, reflections, and rotations.
- Transformations that are not rigid motions are dilations, vertical stretches or compressions, and horizontal stretches or compressions.
Translations

- A translation is sometimes called a slide.
- In a translation, the figure is moved horizontally and/or vertically.
- The orientation of the figure remains the same.
- Connecting the corresponding vertices of the preimage and image will result in a set of parallel lines.

Translating a Figure Given the Horizontal and Vertical Shift

1. Place your pencil on a vertex and count over horizontally the number of units the figure is to be translated.
2. Without lifting your pencil, count vertically the number of units the figure is to be translated.
3. Mark the image vertex on the coordinate plane.
4. Repeat this process for all vertices of the figure.
5. Connect the image vertices.

Reflections

- A reflection creates a mirror image of the original figure over a reflection line.
- A reflection line can pass through the figure, be on the figure, or be outside the figure.
- Reflections are sometimes called flips.
- The orientation of the figure is changed in a reflection.
- In a reflection, the corresponding vertices of the preimage and image are equidistant from the line of reflection, meaning the distance from each vertex to the line of reflection is the same.
- The line of reflection is the perpendicular bisector of the segments that connect the corresponding vertices of the preimage and the image.
Reflecting a Figure over a Given Reflection Line

1. Draw the reflection line on the same coordinate plane as the figure.

2. If the reflection line is vertical, count the number of horizontal units one vertex is from the line and count the same number of units on the opposite side of the line. Place the image vertex there. Repeat this process for all vertices.

3. If the reflection line is horizontal, count the number of vertical units one vertex is from the line and count the same number of units on the opposite side of the line. Place the image vertex there. Repeat this process for all vertices.

4. If the reflection line is diagonal, draw lines from each vertex that are perpendicular to the reflection line extending beyond the line of reflection. Copy each segment from the vertex to the line of reflection onto the perpendicular line on the other side of the reflection line and mark the image vertices.

5. Connect the image vertices.

Rotations

- A rotation moves all points of a figure along a circular arc about a point. Rotations are sometimes called turns.

- In a rotation, the orientation is changed.

- The point of rotation can lie on, inside, or outside the figure, and is the fixed location that the object is turned around.

- The angle of rotation is the measure of the angle created by the preimage vertex to the point of rotation to the image vertex. All of these angles are congruent when a figure is rotated.

- Rotating a figure clockwise moves the figure in a circular arc about the point of rotation in the same direction that the hands move on a clock.

- Rotating a figure counterclockwise moves the figure in a circular arc about the point of rotation in the opposite direction that the hands move on a clock.
Rotating a Figure Given a Point and Angle of Rotation

1. Draw a line from one vertex to the point of rotation.
2. Measure the angle of rotation using a protractor.
3. Draw a ray from the point of rotation extending outward that creates the angle of rotation.
4. Copy the segment connecting the point of rotation to the vertex (created in step 1) onto the ray created in step 3.
5. Mark the endpoint of the copied segment that is not the point of rotation with the letter of the corresponding vertex, followed by a prime mark (‘). This is the first vertex of the rotated figure.
6. Repeat the process for each vertex of the figure.
7. Connect the vertices that have prime marks. This is the rotated figure.

Common Errors/Misconceptions

- creating the angle of rotation in a clockwise direction instead of a counterclockwise direction and vice versa
- reflecting a figure about a line other than the one given
- mistaking a rotation for a reflection
- misidentifying a translation as a reflection or a rotation
Guided Practice 1.4.1

Example 1

Describe the transformation that has taken place in the diagram below.

1. Examine the orientation of the figures to determine if the orientation has changed or stayed the same. Look at the sides of the triangle.

<table>
<thead>
<tr>
<th>Side length</th>
<th>Preimage orientation</th>
<th>Image orientation</th>
</tr>
</thead>
<tbody>
<tr>
<td>Shortest</td>
<td>Bottom of triangle and horizontal</td>
<td>Bottom of triangle and horizontal</td>
</tr>
<tr>
<td>Longest</td>
<td>Right side of triangle going from top left to bottom right</td>
<td>Right side of triangle going from top left to bottom right</td>
</tr>
<tr>
<td>Intermediate</td>
<td>Left side of triangle and vertical</td>
<td>Left side of triangle and vertical</td>
</tr>
</tbody>
</table>

The orientation of the triangles has remained the same.
2. Connect the corresponding vertices with lines.

The lines connecting the corresponding vertices appear to be parallel.

3. Analyze the change in position.

Check the horizontal distance of vertex $A$. To go from $A$ to $A'$ horizontally, the vertex was shifted to the right 6 units. Vertically, vertex $A$ was shifted down 5 units. Check the remaining two vertices. Each vertex slid 6 units to the right and 5 units down.
Example 2
Describe the transformation that has taken place in the diagram below.

1. Examine the orientation of the figures to determine if the orientation has changed or stayed the same. Look at the sides of the figures and pick a reference point. A good reference point is the outer right angle of the figure. From this point, examine the position of the “arms” of the figure.

<table>
<thead>
<tr>
<th>Arm</th>
<th>Preimage orientation</th>
<th>Image orientation</th>
</tr>
</thead>
<tbody>
<tr>
<td>Shorter</td>
<td>Pointing upward from the corner of the figure with a negative slope at the end of the arm</td>
<td>Pointing downward from the corner of the figure with a positive slope at the end of the arm</td>
</tr>
<tr>
<td>Longer</td>
<td>Pointing to the left from the corner of the figure with a positive slope at the end of the arm</td>
<td>Pointing to the left from the corner of the figure with a negative slope at the end of the arm</td>
</tr>
</tbody>
</table>

(continued)
The orientation of the figures has changed. In the preimage, the outer right angle is in the bottom right-hand corner of the figure, with the shorter arm extending upward. In the image, the outer right angle is on the top right-hand side of the figure, with the shorter arm extending down.

Also, compare the slopes of the segments at the end of the longer arm. The slope of the segment at the end of the arm is positive in the preimage, but in the image the slope of the corresponding arm is negative. A similar reversal has occurred with the segment at the end of the shorter arm. In the preimage, the segment at the end of the shorter arm is negative, while in the image the slope is positive.

2. Determine the transformation that has taken place.
Since the orientation has changed, the transformation is either a reflection or a rotation. Since the orientation of the image is the mirror image of the preimage, the transformation is a reflection. The figure has been flipped over a line.

3. Determine the line of reflection.
Connect some of the corresponding vertices of the figure. Choose one of the segments you created and construct the perpendicular bisector of the segment. Verify that this is the perpendicular bisector for all segments joining the corresponding vertices. This is the line of reflection.

The line of reflection for this figure is $y = -1$. 

---

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CCGPS Analytic Geometry Teacher Resource
Example 3
Describe the transformation that has taken place in the diagram below.

1. Examine the orientation of the figures to determine if the orientation has changed or stayed the same.
   Look at the sides of the triangle.

<table>
<thead>
<tr>
<th>Side length</th>
<th>Preimage orientation</th>
<th>Image orientation</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Shortest</strong></td>
<td>Right side of triangle and vertical</td>
<td>Top of triangle and horizontal</td>
</tr>
<tr>
<td><strong>Longest</strong></td>
<td>Diagonal from bottom left to top right of triangle</td>
<td>Diagonal from top left to bottom right of triangle</td>
</tr>
<tr>
<td><strong>Intermediate</strong></td>
<td>Bottom side of triangle and horizontal</td>
<td>Right side of triangle and vertical</td>
</tr>
</tbody>
</table>

The orientation of the triangles has changed.
2. Determine the transformation that has taken place.

Since the orientation has changed, the transformation is either a reflection or a rotation. Since the orientation of the image is NOT the mirror image of the preimage, the transformation is a rotation. The figure has been turned about a point. All angles that are made up of the preimage vertex to the reflection point to the corresponding image vertex are congruent. This means that $\angle ARA' \cong \angle BRB' \cong \angle CRC'$. 

![Diagram](image-url)
Example 4

Rotate the given figure 45° counterclockwise about the origin.

1. Create the angle of rotation for the first vertex.

Connect vertex $A$ and the origin with a line segment. Label the point of reflection $R$. Then, use a protractor to measure a 45° angle. Use the segment from vertex $A$ to the point of rotation $R$ as one side of the angle. Mark a point $X$ at 45°. Draw a ray extending out from point $R$, connecting $R$ and $X$. Copy $\overline{AR}$ onto $\overline{RX}$. Label the endpoint that leads away from the origin $A'$. 
2. Create the angle of rotation for the second vertex.

Connect vertex $B$ and the origin with a line segment. The point of reflection is still $R$. Then, use a protractor to measure a 45° angle. Use the segment from vertex $B$ to the point of rotation $R$ as one side of the angle. Mark a point $Y$ at 45°. Draw a ray extending out from point $R$, connecting $R$ and $Y$. Copy $\overline{BR}$ onto $\overline{RY}$. Label the endpoint that leads away from the origin $B'$. 
3. Create the angle of rotation for the third vertex.

Connect vertex \( C \) and the origin with a line segment. The point of reflection is still \( R \). Then, use a protractor to measure a 45° angle. Use the segment from vertex \( C \) to the point of rotation \( R \) as one side of the angle. Mark a point \( Z \) at 45°. Draw a ray extending out from point \( R \), connecting \( R \) and \( Z \) and continuing outward from \( Z \). Copy \( \overline{CR} \) onto \( \overline{RZ} \). Label the endpoint that leads away from the origin \( C' \).

4. Connect the rotated points.

The connected points \( A', B', \) and \( C' \) form the rotated figure.
Problem-Based Task 1.4.1: Artifacts Recovered and Restored

The Titanic sank on the morning of April 15, 1912, after hitting an iceberg. In recent years, many artifacts from the ship have been recovered and some have been restored. Your job is to coordinate the recovery and restoration of a particular artifact from the Titanic. Your search team will use an unmanned vessel to retrieve the artifact so artists can restore it.

Your team controls the unmanned vessel remotely so it can navigate around objects. The vessel can be translated and rotated, but it cannot touch any of the objects—it must be at least 1 unit away from each item. The point of rotation is at the center of the vessel. The vessel is roughly the shape of a triangle and always points forward when moving. The vessel is currently positioned forward toward Object 3.

To restore the damaged artifact to its original condition, artists will use reflections.

What rigid motions must be used for the unmanned vessel to recover the artifact? What will the artifact look like after it has been restored? Use the following diagrams to answer the questions. The diagram on the left represents a map of the sea floor, including the vessel, the artifact, and objects to avoid. The diagram on the right shows what you expect to recover of the artifact.

![Diagram](image-url)
Problem-Based Task 1.4.1: Artifacts Recovered and Restored

Coaching

a. What rigid motion would move the vessel between Objects 1 and 2?

b. Several rigid motions need to be performed to turn and move the vessel to navigate around Object 3. What are they?

c. What rigid motion(s) can be used to navigate toward the artifact?

d. What rigid motion needs to be performed to restore the artifact?

e. What does the artifact look like after this rigid motion is performed?
Problem-Based Task 1.4.1: Artifacts Recovered and Restored

Coaching Sample Responses

a. What rigid motion would move the vessel between Objects 1 and 2?
   Translating the vessel 8 units down.

b. Several rigid motions need to be performed to turn and move the vessel to navigate around Object 3. What are they?
   First, rotate the vessel 90° counterclockwise. Then translate the vessel 7 units to the right. Rotate the vessel again, but this time rotate it 90° clockwise. Translate the vessel 7 units down.

c. What rigid motion(s) can be used to navigate toward the artifact?
   Depending on where the vessel was left in the last series of motions, answers may vary. From where the vessel stands at this point, the vessel needs to be translated 3 units down and then rotated clockwise 90°. Translate the vessel 4 units to the left.
d. What rigid motion needs to be performed to restore the artifact?

According to the task, artists use reflections to restore missing pieces of artifacts.

e. What does the artifact look like after this rigid motion is performed?

**Recommended Closure Activity**
Select one or more of the essential questions for a class discussion or as a journal entry prompt.
Practice 1.4.1: Describing Rigid Motions and Predicting the Effects

For problems 1–3, describe the rigid motion used to transform each figure. If the transformation is a translation, state the units and direction(s) the figure was transformed. If the transformation is a reflection, state the line of reflection. Write a statement justifying your answer.

1.

2.

3.

continued
For problems 4–10, use the given rigid motion to predict the effect(s) it will have on the given figure.

4. Rotate the given triangle 120° counterclockwise about the origin.

5. Translate the given quadrilateral to the left 6 units and down 5 units.

6. Reflect the given triangle over the line \( x = 3 \).
7. An interior designer wants to rotate the couch in a family room $60^\circ$ about the center of the coffee table, $T$. What will be the final position of the couch?

8. Due to the widening of a road, a house needs to be moved west 8 feet and south 32 feet. The house is shown as $\square ABCD$ in the diagram below. Each square on the grid represents 8 feet. What will be the final position of the house?
9. Artists often paint the reflection of a mountain peak in a nearby body of water. Reflect the mountain over the body of water represented by the dashed line.

10. A ceiling fan has 4 paddles that are separated by a constant angle of rotation. Each paddle has a line of symmetry, as shown in the diagram below. Describe a series of rigid motions that can be performed with the figure below to generate a ceiling fan with 4 paddles. Keep in mind that there are 360° degrees of rotation to bring a figure back onto itself. Draw the figure of the completed fan.
August’s grandparents bought her family a new flat-screen TV as a housewarming present. However, the new TV is too wide to fit into the piece of furniture that was holding the old TV. August proposed that they rotate the TV 30º counterclockwise about point $R$ to slide the TV into the cabinet. She drew a scaled diagram that shows the top view of the TV and TV cabinet. Use the diagram to solve the problems that follow.

1. Rotate the TV counterclockwise 30º about point $R$.

2. Will the rotated TV fit into the cabinet? Justify your answer.
August’s grandparents bought her family a new flat-screen TV as a housewarming present. However, the new TV is too wide to fit into the piece of furniture that was holding the old TV. August proposed that they rotate the TV 30º counterclockwise about point $R$ to slide the TV into the cabinet. She drew a scaled diagram that shows the top view of the TV and TV cabinet.

1. Rotate the TV counterclockwise 30º about point $R$. 

![Diagram of TV and TV cabinet with rotation indicated]
2. Will the rotated TV fit into the cabinet? Justify your answer.

Since the drawing is a scaled representation of the actual figures, use the units provided on the grid and draw the diagonal of the TV cabinet. Use the Pythagorean Theorem to calculate the length of the diagonal.

\[ 4^2 + 7^2 = d^2, \text{ for which } d \text{ is the diagonal} \]
\[ 16 + 49 = d^2 \]
\[ 65 = d^2 \]
\[ \sqrt{65} = d \]

Since distance can only be positive, \( d = \sqrt{65} \approx 8.06 \) units.

Count the length of the TV.

The TV is 8 units. Therefore, the TV should fit into the cabinet on the diagonal. However, the fit will be tight.

**Connection to the Lesson**

- Students will be introduced to the terms and concepts of transformations as rigid and non-rigid motions.
- Students will use the Pythagorean Theorem to determine if rigid motions have occurred.
Introduction

Rigid motions can also be called congruency transformations. A congruency transformation moves a geometric figure but keeps the same size and shape. Preimages and images that are congruent are also said to be isometries. If a figure has undergone a rigid motion or a set of rigid motions, the preimage and image are congruent. When two figures are congruent, they have the same shape and size. Remember that rigid motions are translations, reflections, and rotations. Non-rigid motions are dilations, stretches, and compressions. Non-rigid motions are transformations done to a figure that change the figure’s shape and/or size.

Key Concepts

- To decide if two figures are congruent, determine if the original figure has undergone a rigid motion or set of rigid motions.
- If the figure has undergone only rigid motions (translations, reflections, or rotations), then the figures are congruent.
- If the figure has undergone any non-rigid motions (dilations, stretches, or compressions), then the figures are not congruent. A dilation uses a center point and a scale factor to either enlarge or reduce the figure. A dilation in which the figure becomes smaller can also be called a compression.
- A scale factor is a multiple of the lengths of the sides from one figure to the dilated figure. The scale factor remains constant in a dilation.
- If the scale factor is larger than 1, then the figure is enlarged.
- If the scale factor is between 0 and 1, then the figure is reduced.
- To calculate the scale factor, divide the length of the sides of the image by the lengths of the sides of the preimage.
Lesson 4: Exploring Congruence

- A vertical stretch or compression preserves the horizontal distance of a figure, but changes the vertical distance.
- A horizontal stretch or compression preserves the vertical distance of a figure, but changes the horizontal distance.
- To verify if a figure has undergone a non-rigid motion, compare the lengths of the sides of the figure. If the sides remain congruent, only rigid motions have been performed.
- If the side lengths of a figure have changed, non-rigid motions have occurred.

### Non-Rigid Motions

<table>
<thead>
<tr>
<th>Dilations</th>
<th>Vertical transformations</th>
<th>Horizontal transformations</th>
</tr>
</thead>
<tbody>
<tr>
<td>Enlargement/reduction</td>
<td>Stretch/compression</td>
<td>Stretch/compression</td>
</tr>
</tbody>
</table>

Compare \( \triangle ABC \) with \( \triangle DEF \). The size of each side changes by a constant scale factor. The angle measures have stayed the same.

Compare \( \triangle ABC \) with \( \triangle ADC \). The vertical distance changes by a scale factor. The horizontal distance remains the same. Two of the angles have changed measures.

Compare \( \triangle ABC \) with \( \triangle ABD \). The horizontal distance changes by a scale factor. The vertical distance remains the same. Two of the angles have changed measures.

### Common Errors/Misconceptions
- mistaking a non-rigid motion for a rigid motion and vice versa
- not recognizing that rigid motions preserve shape and size
- not recognizing that it takes only one non-rigid motion to render two figures not congruent
Example 1

Determine if the two figures below are congruent by identifying the transformations that have taken place.

1. Determine the lengths of the sides.

   For the horizontal and vertical legs, count the number of units for the length. For the hypotenuse, use the Pythagorean Theorem, \( a^2 + b^2 = c^2 \), for which \( a \) and \( b \) are the legs and \( c \) is the hypotenuse.

   \[
   \begin{align*}
   AC &= 3 & A'C' &= 3 \\
   CB &= 5 & C'B' &= 5 \\
   AC^2 + CB^2 &= AB^2 & A'C'^2 + C'B'^2 &= A'B'^2 \\
   3^2 + 5^2 &= AB^2 & 3^2 + 5^2 &= A'B'^2 \\
   34 &= AB^2 & 34 &= A'B'^2 \\
   \sqrt{34} &= \sqrt{AB^2} & \sqrt{34} &= \sqrt{A'B'^2} \\
   AB &= \sqrt{34} & A'B' &= \sqrt{34}
   \end{align*}
   \]

   The sides in the first triangle are congruent to the sides of the second triangle. \textit{Note:} When taking the square root of both sides of the equation, reject the negative value since the value is a distance and distance can only be positive.
2. Identify the transformations that have occurred.

The orientation has changed, indicating a rotation or a reflection.

The second triangle is a mirror image of the first, but translated to the right 4 units.

The triangle has undergone rigid motions: reflection and translation.

3. State the conclusion.

The triangle has undergone two rigid motions: reflection and translation. Rigid motions preserve size and shape. The triangles are congruent.
Example 2

Determine if the two figures below are congruent by identifying the transformations that have taken place.

1. Determine the lengths of the sides.

For the horizontal and vertical legs, count the number of units for the length. For the hypotenuse, use the Pythagorean Theorem, \( a^2 + b^2 = c^2 \), for which \( a \) and \( b \) are the legs and \( c \) is the hypotenuse.

\[
\begin{align*}
AB &= 3 \\
AC &= 4 \\
A'B' &= 6 \\
A'C' &= 8 \\
AB^2 + AC^2 &= CB^2 \\
A'B'^2 + A'C'^2 &= C'B'^2 \\
3^2 + 4^2 &= CB^2 \\
6^2 + 8^2 &= C'B'^2 \\
25 &= CB^2 \\
100 &= C'B'^2 \\
\sqrt{25} &= \sqrt{CB^2} \\
\sqrt{100} &= \sqrt{C'B'^2} \\
CB &= \sqrt{25} \\
C'B' &= \sqrt{100} \\
CB &= 5 \\
C'B' &= 10
\end{align*}
\]

The sides in the first triangle are not congruent to the sides of the second triangle. They are not the same size.
2. Identify the transformations that have occurred.

The orientation has stayed the same, indicating translation, dilation, stretching, or compression. The vertical and horizontal distances have changed. This could indicate a dilation.

3. Calculate the scale factor of the changes in the side lengths.

Divide the image side lengths by the preimage side lengths.

\[
\frac{A'B'}{AB} = \frac{6}{3} = 2
\]
\[
\frac{A'C'}{AC} = \frac{8}{4} = 2
\]
\[
\frac{C'B'}{CB} = \frac{10}{5} = 2
\]

The scale factor is constant between each pair of sides in the preimage and image. The scale factor is 2, indicating a dilation. Since the scale factor is greater than 1, this is an enlargement.

4. State the conclusion.

The triangle has undergone at least one non-rigid motion: a dilation. Specifically, the dilation is an enlargement with a scale factor of 2. The triangles are not congruent because dilation does not preserve the size of the original triangle.
Example 3

Determine if the two figures below are congruent by identifying the transformations that have taken place.

1. Determine the lengths of the sides.

   For the horizontal and vertical sides, count the number of units for the length.

   \[
   \begin{align*}
   AB &= 6 & A'B' &= 4.5 \\
   BC &= 4 & B'C' &= 4 \\
   CD &= 6 & C'D' &= 4.5 \\
   DA &= 4 & D'A' &= 4
   \end{align*}
   \]

   Two of the sides in the first rectangle are not congruent to two of the sides of the second rectangle. Two sides are congruent in the first and second rectangles.
2. Identify the transformations that have occurred.

The orientation has changed, and two side lengths have changed. The change in side length indicates at least one non-rigid motion has occurred. Since not all pairs of sides have changed in length, the non-rigid motion must be a horizontal or vertical stretch or compression.

The image has been reflected since \( \overline{BC} \) lies at the top of the preimage and \( \overline{B'C'} \) lies at the bottom of the image. Reflections are rigid motions. However, one non-rigid motion makes the figures not congruent. A non-rigid motion has occurred since not all the sides in the image are congruent to the sides in the preimage.

The vertical lengths have changed, while the horizontal lengths have remained the same. This means the transformation must be a vertical transformation.
3. Calculate the scale factor of the change in the vertical sides.

Divide the image side lengths by the preimage side lengths.

\[
\frac{A'B'}{AB} = \frac{4.5}{6} = 0.75
\]

\[
\frac{C'D'}{CD} = \frac{4.5}{6} = 0.75
\]

The vertical sides have a scale factor of 0.75. The scale factor is between 0 and 1, indicating compression. Since only the vertical sides changed, this is a vertical compression.

4. State the conclusion.

The vertical sides of the rectangle have undergone at least one non-rigid transformation of a vertical compression. The vertical sides have been reduced by a scale factor of 0.75. Since a non-rigid motion occurred, the figures are not congruent.
Example 4

Determine if the two figures below are congruent by identifying the transformations that have taken place.

1. Determine the lengths of the sides.

   For the horizontal and vertical sides, count the number of units for the length. For the hypotenuse, use the Pythagorean Theorem, \(a^2 + b^2 = c^2\), for which \(a\) and \(b\) are the legs and \(c\) is the hypotenuse.

   \[
   BC = 11 \quad B'C' = 11 \\
   CA = 7 \quad C'A' = 7 \\
   BC^2 + CA^2 = AB^2 \quad B'C'^2 + C'A'^2 = A'B'^2 \\
   11^2 + 7^2 = AB^2 \quad 11^2 + 7^2 = A'B'^2 \\
   170 = AB^2 \quad 170 = A'B'^2 \\
   \sqrt{170} = \sqrt{AB^2} \quad \sqrt{170} = \sqrt{A'B'^2} \\
   AB = \sqrt{170} \quad A'B' = \sqrt{170}
   \]

   The sides of the first triangle are congruent to the sides of the second triangle.
2. Identify the transformations that have occurred.

   The orientation has changed and all side lengths have stayed the same. This indicates a reflection or a rotation. The preimage and image are not mirror images of each other. Therefore, the transformation that occurred is a rotation.

3. State the conclusion.

   Rotations are rigid motions and rigid motions preserve size and shape. The two figures are congruent.
Problem-Based Task 1.4.2: Architectural Planning

Architects use scale drawings to map out floor plans of houses. In the floor plan below, $\frac{1}{4}$ inch equals 24 inches in the actual house. An architect is creating a housing development of 10 houses. She plans to use one basic floor plan and then use rigid motions to adjust the rooms to create a different look for the 10 houses. Using the floor plan provided below, create at least two different floor plans using only rigid motions. Describe the non-rigid motion used to build an actual house based on the floor plan.
Problem-Based Task 1.4.2: Architectural Planning

Coaching

a. What transformation can be used to reverse the front rooms with the back rooms of the house?

b. How do you apply this transformation?

c. What does the result look like? Draw it.

d. Are the floor plans congruent? Explain.

e. What transformation can be used to reverse the rooms on the left with the rooms on the right?

f. How do you apply this transformation?

g. What does the result look like? Draw it.

h. Are the floor plans congruent? Explain.

i. What non-rigid motion is used to scale a figure?

j. What is the scale factor from the floor plan to the actual house?

k. Are the house and the floor plan congruent? Explain.
Problem-Based Task 1.4.2: Architectural Planning

Coaching Sample Responses

a. What transformation can be used to reverse the front rooms with the back rooms of the house?

A reflection can be used to create a mirror image, which would switch the front rooms with the back rooms.

b. How do you apply this transformation?

Draw a horizontal line at the front of the house plan and reflect the floor plan over that line. Alternatively, you could draw a horizontal line at the back of the house.

c. What does the result look like? Draw it.

Answers may vary depending on where you draw the line of reflection. Pictured below is the floor plan reflected over a horizontal line at the front of the house.
d. Are the floor plans congruent? Explain.

A reflection is a rigid motion and rigid motions preserve shape and size. The floor plans are congruent.

e. What transformation can be used to reverse the rooms on the left with the rooms on the right?

A reflection can be used to create a mirror image, which would switch the rooms on the left with the rooms on the right.

f. How do you apply this transformation?

Draw a vertical line to the left of the house plan and reflect the floor plan over that line. Alternatively, draw a vertical line to the right of the house.

g. What does the result look like? Draw it.

Answers may vary depending on where you draw the line of reflection. Pictured below is the floor plan reflected over a vertical line to the left of the plan.
Lesson 4: Exploring Congruence

h. Are the floor plans congruent? Explain.
   
   A reflection is a rigid motion and rigid motions preserve shape and size. The floor plans are congruent.

i. What non-rigid motion is used to scale a figure?
   
   A dilation scales a figure in both directions. If the scale factor is between 0 and 1, then the dilation is a reduction. If the scale factor is greater than 1, then the dilation is an enlargement.

j. What is the scale factor from the floor plan to the actual house?
   
   \[
   \frac{\text{house in inches}}{\text{floor plan in inches}} = \frac{24}{1} = 24 \cdot \frac{4}{1} = 96
   \]
   
   The scale factor is 96.

k. Are the house and the floor plan congruent? Explain.
   
   No, they are not congruent, because the sizes are not the same. A dilation has occurred. Specifically, an enlargement occurs from the floor plan to the house, with a scale factor of 96.

**Recommended Closure Activity**

Select one or more of the essential questions for a class discussion or as a journal entry prompt.
Practice 1.4.2: Defining Congruence in Terms of Rigid Motions

Determine if the two given figures are congruent by identifying the transformation(s) that occurred. State whether each transformation is rigid or non-rigid.

1.

2.

3.

continued
Lesson 4: Exploring Congruence

4.

5.

6.

continued
7. Xander is rearranging the setup in the school gym for a large presentation. He has to move the speaker to make room for chairs. The diagram of how the speaker is transformed is pictured below. Describe the transformations that have taken place and determine whether the figures are congruent in terms of rigid and non-rigid motions.

8. A corner cabinet sits in the dining room as pictured below, where the gray line represents the walls of the room. The new location of the corner cabinet is seen as the triangle labeled $A'B'C'$ in the diagram. Describe the transformations that have taken place and determine whether the figures representing the corner cabinet are congruent in terms of rigid and non-rigid motions.
9. The picture frame pictured below can be thought of as a square inside another square. Describe the transformations that have taken place and determine whether the squares are congruent in terms of rigid and non-rigid motions.

10. A truss is a structure used in building bridges. The bridge truss pictured below is made up of 5 triangles. Describe the transformations that have taken place and determine whether the triangles are congruent in terms of rigid and non-rigid motions.
UNIT 1 • SIMILARITY, CONGRUENCE, AND PROOFS

Lesson 5: Congruent Triangles

Common Core Georgia Performance Standards

MCC9–12.G.CO.7
MCC9–12.G.CO.8

Essential Questions

1. Why is it important to know how to mark congruence on a diagram?
2. What does it mean if two triangles are congruent?
3. If two triangles have two sides and one angle that are equivalent, can congruence be determined?
4. How many equivalent measures are needed to determine if triangles are congruent?

WORDS TO KNOW

Angle-Side-Angle (ASA) Congruence Statement

If two angles and the included side of one triangle are congruent to two angles and the included side of another triangle, then the two triangles are congruent.

congruent angles
two angles that have the same measure

congruent sides
two sides that have the same length

congruent triangles
triangles having the same angle measures and side lengths

corresponding angles
a pair of angles in a similar position

Corresponding Parts of Congruent Triangles are Congruent (CPCTC)

If two or more triangles are proven congruent, then all of their corresponding parts are congruent as well.

corresponding sides
the sides of two figures that lie in the same position relative to the figures

included angle
the angle between two sides

included side
the side between two angles of a triangle

postulate
a true statement that does not require a proof

rigid motion
a transformation done to an object that maintains the object’s shape and size or its segment lengths and angle measures
Side-Angle-Side (SAS) Congruence Statement
If two sides and the included angle of one triangle are congruent to two sides and the included angle of another triangle, then the two triangles are congruent.

Side-Side-Side (SSS) Congruence Statement
If three sides of one triangle are congruent to three sides of another triangle, then the two triangles are congruent.

Recommended Resources

- Illuminations. “Congruence Theorems.”
  http://www.walch.com/rr/00013
  This site allows users to investigate congruence postulates by manipulating parts of a triangle.

- Math Open Reference. “Congruent Triangles.”
  http://www.walch.com/rr/00014
  This site provides an explanation of congruent triangles, as well as interactive graphics that demonstrate how congruent triangles remain congruent when different transformations are applied.

- Math Warehouse. “Corresponding Sides and Angles: Identify Corresponding Parts.”
  http://www.walch.com/rr/00015
  This site explains how to identify corresponding parts of a triangle, and provides interactive questions and answers.

  http://www.walch.com/rr/00016
  This site allows users to construct congruent triangles according to each of the congruence postulates.
Lesson 1.5.1: Triangle Congruency

Warm-Up 1.5.1

Cutting mats used in crafting are similar to coordinate planes, except that the axes are labeled with inches or centimeters, rather than positive and negative numbers. Juliet is in the middle of a sewing project and has laid out two congruent pieces of fabric on her cutting mat.

1. What is the series of transformations that has taken place between Piece 1 and Piece 2?
Juliet takes Piece 1 off the mat. She places a third piece of fabric on the cutting mat.

2. What is the series of transformations that has taken place between Piece 2 and Piece 3?

3. Is Piece 3 congruent to Piece 1? Explain your reasoning.
Lesson 1.5.1: Triangle Congruency

Common Core Georgia Performance Standard

MCC9–12.G.CO.7

Warm-Up 1.5.1 Debrief

Cutting mats used in crafting are similar to coordinate planes, except that the axes are labeled with inches or centimeters, rather than positive and negative numbers. Juliet is in the middle of a sewing project and has laid out two congruent pieces of fabric on her cutting mat.

1. What is the series of transformations that has taken place between Piece 1 and Piece 2?

   Piece 1 was slid, or translated, to the right 6 units.
   Piece 1 was also translated down 3 units.
   The position of Piece 2 is the result of a horizontal translation of 6 units to the right and a vertical translation of 3 units down.
Juliet takes Piece 1 off the mat. She places a third piece of fabric on the cutting mat.

2. What is the series of transformations that has taken place between Piece 2 and Piece 3?
   Piece 3 is the mirror image of Piece 2.
   The position of Piece 3 is the reflection of Piece 2 over a vertical line.

3. Is Piece 3 congruent to Piece 1? Explain your reasoning.
   Piece 3 is congruent to Piece 1.
   A translation is a congruency transformation, so Piece 1 is congruent to Piece 2. A reflection is also a congruency transformation, so Piece 3 is congruent to Piece 2. A series of congruency transformations results in a congruent figure, so Piece 3 is congruent to Piece 1.

**Connection to the Lesson**
- Students will use the definition of congruence in terms of rigid motions to identify congruent parts of triangles.
Introduction

If a rigid motion or a series of rigid motions, including translations, rotations, or reflections, is performed on a triangle, then the transformed triangle is congruent to the original. When two triangles are congruent, the corresponding angles have the same measures and the corresponding sides have the same lengths. It is possible to determine whether triangles are congruent based on the angle measures and lengths of the sides of the triangles.

Key Concepts

- To determine whether two triangles are congruent, you must observe the angle measures and side lengths of the triangles.
- When a triangle is transformed by a series of rigid motions, the angles are images of each other and are called corresponding angles.
- Corresponding angles are a pair of angles in a similar position.
- If two triangles are congruent, then any pair of corresponding angles is also congruent.
- When a triangle is transformed by a series of rigid motions, the sides are also images of each other and are called corresponding sides.
- Corresponding sides are the sides of two figures that lie in the same position relative to the figure.
- If two triangles are congruent, then any pair of corresponding sides is also congruent.
- Congruent triangles have three pairs of corresponding angles and three pairs of corresponding sides, for a total of six pairs of corresponding parts.
- If two or more triangles are proven congruent, then all of their corresponding parts are congruent as well. This postulate is known as **Corresponding Parts of Congruent Triangles are Congruent (CPCTC)**. A **postulate** is a true statement that does not require a proof.
• The corresponding angles and sides can be determined by the order of the letters.

• If $\triangle ABC$ is congruent to $\triangle DEF$, the angles of the two triangles correspond in the same order as they are named.

• Use the symbol $\rightarrow$ to show that two parts are corresponding.
  
  Angle $A \rightarrow$ Angle $D$; they are equivalent.
  Angle $B \rightarrow$ Angle $E$; they are equivalent.
  Angle $C \rightarrow$ Angle $F$; they are equivalent.

• The corresponding angles are used to name the corresponding sides.
  
  $\overline{AB} \rightarrow \overline{DE}$; they are equivalent.
  $\overline{BC} \rightarrow \overline{EF}$; they are equivalent.
  $\overline{AC} \rightarrow \overline{DF}$; they are equivalent.

• Observe the diagrams of $\triangle ABC$ and $\triangle DEF$.

  $\triangle ABC \cong \triangle DEF$

  $\angle A \cong \angle D$
  $\angle B \cong \angle E$
  $\angle C \cong \angle F$

  $\overline{AB} \cong \overline{DE}$
  $\overline{BC} \cong \overline{EF}$
  $\overline{AC} \cong \overline{DF}$

• By observing the angles and sides of two triangles, it is possible to determine if the triangles are congruent.

• Two triangles are congruent if the corresponding angles are congruent and corresponding sides are congruent.

• Notice the number of tick marks on each side of the triangles in the diagram.

• The tick marks show the sides that are congruent.
UNIT 1 • SIMILARITY, CONGRUENCE, AND PROOFS
Lesson 5: Congruent Triangles

Instruction

- Compare the number of tick marks on the sides of $\triangle ABC$ to the tick marks on the sides of $\triangle DEF$.
- Match the number of tick marks on one side of one triangle to the side with the same number of tick marks on the second triangle.
  - $\overline{AB}$ and $\overline{DE}$ each have one tick mark, so the two sides are congruent.
  - $\overline{BC}$ and $\overline{EF}$ each have two tick marks, so the two sides are congruent.
  - $\overline{AC}$ and $\overline{DF}$ each have three tick marks, so the two sides are congruent.
- The arcs on the angles show the angles that are congruent.
- Compare the number of arcs on the angles of $\triangle ABC$ to the number of arcs on the angles of $\triangle DEF$.
- Match the arcs on one angle of one triangle to the angle with the same number of arcs on the second triangle.
  - $\angle A$ and $\angle D$ each have one arc, so the two angles are congruent.
  - $\angle B$ and $\angle E$ each have two arcs, so the two angles are congruent.
  - $\angle C$ and $\angle F$ each have three arcs, so the two angles are congruent.
- If the sides and angles are not labeled as congruent, you can use a ruler and protractor or construction methods to measure each of the angles and sides.

Common Errors/Misconceptions
- incorrectly identifying corresponding parts of triangles
- assuming corresponding parts indicate congruent parts
- assuming alphabetical order indicates congruence
Guided Practice 1.5.1

Example 1

Use corresponding parts to identify the congruent triangles.

1. Match the number of tick marks to identify the corresponding congruent sides.
   - $\overline{RV}$ and $\overline{JM}$ each have one tick mark; therefore, they are corresponding and congruent.
   - $\overline{VA}$ and $\overline{MT}$ each have two tick marks; therefore, they are corresponding and congruent.
   - $\overline{RA}$ and $\overline{JT}$ each have three tick marks; therefore, they are corresponding and congruent.
2. Match the number of arcs to identify the corresponding congruent angles.

\( \angle R \) and \( \angle J \) each have one arc; therefore, the two angles are corresponding and congruent.

\( \angle V \) and \( \angle M \) each have two arcs; therefore, the two angles are corresponding and congruent.

\( \angle A \) and \( \angle T \) each have three arcs; therefore, the two angles are corresponding and congruent.

3. Order the congruent angles to name the congruent triangles.

\( \triangle RVA \) is congruent to \( \triangle JMT \), or \( \triangle RVA \cong \triangle JMT \).

It is also possible to identify the congruent triangles as \( \triangle VAR \cong \triangle MTJ \), or even \( \triangle ARV \cong \triangle TJM \); whatever order chosen, it is important that the order in which the vertices are listed in the first triangle matches the congruency of the vertices in the second triangle.

For instance, it is not appropriate to say that \( \triangle RVA \) is congruent to \( \triangle MJT \) because \( \angle R \) is not congruent to \( \angle M \).
Example 2

\[ \triangle BDF \cong \triangle HJL \]

Name the corresponding angles and sides of the congruent triangles.

1. Identify the congruent angles.
   
   The names of the triangles indicate the angles that are corresponding and congruent. Begin with the first letter of each name.
   
   Identify the first set of congruent angles.
   
   \( \angle B \) is congruent to \( \angle H \).
   
   Identify the second set of congruent angles.
   
   \( \angle D \) is congruent to \( \angle J \).
   
   Identify the third set of congruent angles.
   
   \( \angle F \) is congruent to \( \angle L \).

2. Identify the congruent sides.
   
   The names of the triangles indicate the sides that are corresponding and congruent. Begin with the first two letters of each name.
   
   Identify the first set of congruent sides.
   
   \( \overline{BD} \) is congruent to \( \overline{HJ} \).
   
   Identify the second set of congruent sides.
   
   \( \overline{DF} \) is congruent to \( \overline{JL} \).
   
   Identify the third set of congruent sides.
   
   \( \overline{BF} \) is congruent to \( \overline{HL} \).
Example 3

Use construction tools to determine if the triangles are congruent. If they are, name the congruent triangles and corresponding angles and sides.

1. Use a compass to compare the length of each side.

   Begin with the shortest sides, \( PR \) and \( UT \).

   Set the sharp point of the compass on point \( P \) and extend the pencil of the compass to point \( R \).

   Without changing the compass setting, set the sharp point of the compass on point \( U \) and extend the pencil of the compass to point \( T \).

   The compass lengths match, so the length of \( UT \) is equal to \( PR \); therefore, the two sides are congruent.

   Compare the longest sides, \( PQ \) and \( US \).

   Set the sharp point of the compass on point \( P \) and extend the pencil of the compass to point \( Q \).

   Without changing the compass setting, set the sharp point of the compass on point \( U \) and extend the pencil of the compass to point \( S \).

   The compass lengths match, so the length of \( US \) is equal to \( PQ \); therefore, the two sides are congruent.

(continued)
Compare the last pair of sides, $QR$ and $ST$.

Set the sharp point of the compass on point $Q$ and extend the pencil of the compass to point $R$.

Without changing the compass setting, set the sharp point of the compass on point $S$ and extend the pencil of the compass to point $T$.

The compass lengths match, so the length of $ST$ is equal to $QR$; therefore, the two sides are congruent.

2. Use a compass to compare the measure of each angle.

Begin with the largest angles, $\angle R$ and $\angle T$.

Set the sharp point of the compass on point $R$ and create a large arc through both sides of $\angle R$.

Without adjusting the compass setting, set the sharp point on point $T$ and create a large arc through both sides of $\angle T$.

Set the sharp point of the compass on one point of intersection and open it so it touches the second point of intersection.

Use this setting to compare the distance between the two points of intersection on the second triangle.

The measure of $\angle R$ is equal to the measure of $\angle T$; therefore, the two angles are congruent.

The measure of $\angle Q$ is equal to the measure of $\angle S$; therefore, the two angles are congruent.

The measure of $\angle P$ is equal to the measure of $\angle U$; therefore, the two angles are congruent.

3. Summarize your findings.

The corresponding and congruent sides include:

- $\overline{PR} \cong \overline{UT}$
- $\overline{PQ} \cong \overline{US}$
- $\overline{QR} \cong \overline{ST}$

The corresponding and congruent angles include:

- $\angle R \cong \angle T$
- $\angle Q \cong \angle S$
- $\angle P \cong \angle U$

Therefore, $\triangle RQP \cong \triangle TSU$. 
Problem-Based Task 1.5.1: Stained Glass Pattern, Part I

Mary creates stained glass art. She is in the planning stages of creating a new piece and has found a pattern she really likes. Mary studies the pattern to determine which triangles in the pattern are congruent, so that she can cut the correct size pieces of glass. Pictured below is a portion of the pattern. Use the pattern and the information that follows to determine which triangles are congruent. How could Mary use this information to help plan her project?

- $\square ABCD$ and $\square CFGH$ are squares. Each diagonal of a square bisects an opposite pair of angles.
- $\square BEFC$ and $\square DCHI$ are rhombuses. The diagonals of a rhombus bisect the opposite pairs of angles. Remember that opposite pairs of angles are congruent.
- $\overline{EC} \cong \overline{BC}$
Problem-Based Task 1.5.1: Stained Glass Pattern, Part I

Coaching

a. What are the properties of the sides and angles of a square?

b. What are the properties of the sides and angles of a rhombus?

c. Which sides in the pattern are congruent?

d. How do you indicate on a diagram that sides are congruent?

e. Mark the congruent sides on the diagram.

f. Which angles are congruent?

g. How do you indicate on a diagram that angles are congruent?

h. Mark congruent angles on the diagram.

i. Name one pair of congruent triangles. Justify your answer by indicating which corresponding parts are congruent.

j. Name a second pair of congruent triangles. Justify your answer by indicating which corresponding parts are congruent.

k. Name a third pair of congruent triangles. Justify your answer by indicating which corresponding parts are congruent.

l. Name a fourth pair of congruent triangles. Justify your answer by indicating which corresponding parts are congruent.

m. Are any pairs of triangles that you identified congruent? If so, name them.

n. How can Mary use this information to plan her project?
Problem-Based Task 1.5.1: Stained Glass Pattern, Part I

Coaching Sample Responses

a. What are the properties of the sides and angles of a square?
   All sides are congruent and all angles are congruent.

b. What are the properties of the sides and angles of a rhombus?
   All sides are congruent and opposite angles are congruent.

c. Which sides in the pattern are congruent and why?
   \( \overline{AB} \cong \overline{BC} \cong \overline{CD} \cong \overline{DA} \) because \( \square ABCD \) is a square.
   \( \overline{BC} \cong \overline{BE} \cong \overline{EF} \cong \overline{FC} \) because \( \square BEFC \) is a rhombus.
   \( \overline{FC} \cong \overline{FG} \cong \overline{GH} \cong \overline{HC} \) because \( \square CFGH \) is a square.
   \( \overline{HC} \cong \overline{HI} \cong \overline{ID} \cong \overline{DC} \) because \( \square DCHI \) is a rhombus.
   \( \overline{EC} \cong \overline{BC} \) because this was given.

d. How do you indicate on a diagram that sides are congruent?
   Use the same number of tick marks on the sides that are congruent.

e. Mark the congruent sides on the diagram.
f. Which angles are congruent and why?

\( \angle DAB \cong \angle HGF \cong \angle DCB \cong \angle HCF \) because these are the vertices of the squares; the measure of the vertex of a square is always 90°.

\( \angle ABD \cong \angle DBC \cong \angle ADB \cong \angle BDC \) because the diagonal of a square bisects the opposite pairs of angles.

\( \angle CHF \cong \angle GHF \cong \angle CFH \cong \angle GHF \) because the diagonal of a square bisects the opposite pairs of angles.

\( \angle CBE \cong \angle EFC \) because these are the opposite vertices of a rhombus; opposite vertices of a rhombus are congruent.

\( \angle BCE \cong \angle ECF \cong \angle BEC \cong \angle FEC \) because the diagonal of a rhombus bisects the opposite pairs of angles.

\( \angle IDC \cong \angle IHC \) because these are the opposite vertices of a rhombus; opposite vertices of a rhombus are congruent.

\( \angle DIC \cong \angle HIC \cong \angle DCI \cong \angle HCI \) because the diagonal of a rhombus bisects the opposite pairs of angles.

g. How do you indicate on a diagram that angles are congruent?

Use the same number of arcs on the angles that are congruent.

h. Mark congruent angles on the diagram.
i. Name one pair of congruent triangles. Justify your answer by indicating which corresponding parts are congruent.

\[ \triangle ABD \cong \triangle CBD \]

\[ \angle DAB \cong \angle DCB \]

\[ \angle ADB \cong \angle CDB \]

\[ \angle ABD \cong \angle CBD \]

\[ AB \cong CB \]

\[ AD \cong CD \]

\[ BD \cong BD \]

j. Name a second pair of congruent triangles. Justify your answer by indicating which corresponding parts are congruent.

\[ \triangle HCF \cong \triangle HGF \]

\[ \angle HCF \cong \angle HGF \]

\[ \angle CHF \cong \angle GHF \]

\[ \angle CFH \cong \angle GFH \]

\[ HC \cong HG \]

\[ CF \cong GF \]

\[ HF \cong HF \]

k. Name a third pair of congruent triangles. Justify your answer by indicating which corresponding parts are congruent.

\[ \triangle CBE \cong \triangle CFE \]

\[ \angle CBE \cong \angle CFE \]

\[ \angle BEC \cong \angle FEC \]

\[ \angle ECB \cong \angle ECF \]

\[ CB \cong CF \]

\[ BE \cong FE \]

\[ CE \cong CE \]
1. Name a fourth pair of congruent triangles. Justify your answer by indicating which corresponding parts are congruent.

   $\triangle IDC \cong \triangle IHC$

   $\angle IDC \cong \angle IHC$

   $\angle DIC \cong \angle HIC$

   $\angle DCI \cong \angle HCI$

   $\overline{ID} \cong \overline{IH}$

   $\overline{DC} \cong \overline{HC}$

   $\overline{IC} \cong \overline{IC}$

m. Are any pairs of triangles that you identified congruent? If so, name them.

   Yes, $\triangle ABD \cong \triangle CBD \cong \triangle HCF \cong \triangle HGF$.

n. How can Mary use this information to plan her project?

   Mary can use the congruent triangles to create a template for her needed cuts.
   She can then verify that the triangles she has cut will fit her pattern if the cut pieces are congruent.

**Recommended Closure Activity**

Select one or more of the essential questions for a class discussion or as a journal entry prompt.
Practice 1.5.1: Triangle Congruency

Use the diagrams to correctly name each set of congruent triangles according to their corresponding parts.

1. 

2. 

3. 

continued
Name the corresponding angles and sides for each pair of congruent triangles.

4. $\triangle QRS \cong \triangle WXY$

5. $\triangle AFH \cong \triangle CGJ$

6. $\triangle LPQ \cong \triangle HJK$

Use a ruler and a protractor or construction tools to determine if the triangles are congruent. If they are, name the congruent triangles and their corresponding angles and sides.

7. An architect has two versions of a blueprint. Both blueprints contain a ramp. Are the ramps congruent?

8. A recent delivery to a construction site included several trusses for a new roof structure. Two of the trusses are shown below. Are the trusses congruent?
9. Dave is making a blanket out of his old band shirts and has pre-cut several pieces of fabric. Are the T-shirt pieces congruent?

10. A tile installer wants to replace a broken tile with a scrap piece he has from a recent job. Is the scrap piece of tile congruent to the tile that needs replacement?
Lesson 1.5.2: Explaining ASA, SAS, and SSS

Warm-Up 1.5.2

The Great Pyramid of Giza is the oldest and largest of the three pyramids near Giza, Egypt. The pyramid has a square base and four congruent triangular faces.

1. Is it possible to determine if the sides of the triangular faces are congruent? Explain your reasoning.

2. Is it possible to determine if the angles of the triangular faces are congruent? Explain your reasoning.
Lesson 1.5.2: Explaining ASA, SAS, and SSS

Common Core Georgia Performance Standard

MCC9–12.G.CO.8

Warm-Up 1.5.2 Debrief

The Great Pyramid of Giza is the oldest and largest of the three pyramids near Giza, Egypt. The pyramid has a square base and four congruent triangular faces.

1. Is it possible to determine if the sides of the triangular faces are congruent? Explain your reasoning.

Yes, it is possible to determine if the sides of the triangular faces are congruent.

It is known that the base of the pyramid is a square; therefore, $\overline{AB} \cong \overline{BC} \cong \overline{CD} \cong \overline{DA}$.

Also, it is given that the triangular faces are congruent; therefore, the corresponding parts of the triangles are also congruent, so $\overline{AE} \cong \overline{BE} \cong \overline{CE} \cong \overline{DE}$.

2. Is it possible to determine if the angles of the triangular faces are congruent? Explain your reasoning.

Yes, it is possible to determine if the angles of the triangular faces are congruent.

The corresponding angles of the congruent triangles are congruent; therefore, $\angle EBA \cong \angle EAD \cong \angle EDC \cong \angle ECB$. Also, $\angle EAB \cong \angle EBC \cong \angle ECD \cong \angle EDA$.

Connection to the Lesson

- Students will use corresponding parts to determine if triangles are congruent.
Introduction
When a series of rigid motions is performed on a triangle, the result is a congruent triangle. When triangles are congruent, the corresponding parts of the triangles are also congruent. It is also true that if the corresponding parts of two triangles are congruent, then the triangles are congruent. It is possible to determine if triangles are congruent by measuring and comparing each angle and side, but this can take time. There is a set of congruence criteria that lets us determine whether triangles are congruent with less information.

Key Concepts
- The criteria for triangle congruence, known as triangle congruence statements, provide the least amount of information needed to determine if two triangles are congruent.
- Each congruence statement refers to the corresponding parts of the triangles.
- By looking at the information about each triangle, you can determine whether the triangles are congruent.
- The **Side-Side-Side (SSS) Congruence Statement** states that if three sides of one triangle are congruent to three sides of another triangle, then the two triangles are congruent.
- If it is known that the corresponding sides are congruent, it is understood that the corresponding angles are also congruent.
- The **Side-Angle-Side (SAS) Congruence Statement** states that if two sides and the included angle of one triangle are congruent to two sides and the included angle of another triangle, then the two triangles are congruent.
• The **included angle** is the angle that is between the two congruent sides.

<table>
<thead>
<tr>
<th>Included angle</th>
<th>Non-included angle</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\angle A$ is included between $\overline{CA}$ and $\overline{AB}$.</td>
<td>$\angle B$ is NOT included between $\overline{CA}$ and $\overline{AB}$.</td>
</tr>
<tr>
<td>$\angle D$ is included between $\overline{FD}$ and $\overline{DE}$.</td>
<td>$\angle E$ is NOT included between $\overline{FD}$ and $\overline{DE}$.</td>
</tr>
</tbody>
</table>

• The **Angle-Side-Angle Congruence Statement**, or **ASA**, states that if two angles and the included side of one triangle are congruent to two angles and the included side of another triangle, then the two triangles are congruent.

• The **included side** is the side that is between the two congruent angles.

<table>
<thead>
<tr>
<th>Included side</th>
<th>Non-included side</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\overline{AC}$ is included between $\angle C$ and $\angle A$.</td>
<td>$\overline{CB}$ is NOT included between $\angle C$ and $\angle A$.</td>
</tr>
<tr>
<td>$\overline{FD}$ is included between $\angle F$ and $\angle D$.</td>
<td>$\overline{FE}$ is NOT included between $\angle F$ and $\angle D$.</td>
</tr>
</tbody>
</table>

• A fourth congruence statement, angle-angle-side (AAS), states that if two angles and a non-included side of one triangle are congruent to the corresponding two angles and side of a second triangle, then the triangles are congruent.

• This lesson will focus on the first three congruence statements: SSS, SAS, and ASA.
The following diagram compares these three congruence statements.

<table>
<thead>
<tr>
<th>Side-Side-Side (SSS)</th>
<th>Side-Angle-Side (SAS)</th>
<th>Angle-Side-Angle (ASA)</th>
</tr>
</thead>
<tbody>
<tr>
<td>ΔABC ≌ ΔXYZ</td>
<td>ΔDEF ≌ ΔTVW</td>
<td>ΔGHJ ≌ ΔQRS</td>
</tr>
</tbody>
</table>

**Common Errors/Misconceptions**
- misidentifying included sides and angles, resulting in the wrong congruence statement
- misreading congruency symbols of triangles
- changing the order of named triangles, causing parts to be incorrectly interpreted as congruent
Example 1

Determine which congruence statement, if any, can be used to show that $\triangle PQR$ and $\triangle STU$ are congruent.

1. Determine which components of the triangles are congruent.

   According to the diagram, $RP \cong US$, $PQ \cong ST$, and $TU \cong QR$.

   Corresponding side lengths of the two triangles are identified as congruent.

2. Determine if this information is enough to state that all six corresponding parts of the two triangles are congruent.

   It is given that all side lengths of the two triangles are congruent; therefore, all their angles are also congruent.

   Because all six corresponding parts of the two triangles are congruent, then the two triangles are congruent.

3. Summarize your findings.

   $\triangle PQR \cong \triangle STU$ because of the Side-Side-Side (SSS) Congruence Statement.
Example 2
Determine which congruence statement, if any, can be used to show that $\triangle ABC$ and $\triangle DEF$ are congruent.

1. Determine which components of the triangles are congruent.

   According to the diagram, $\overline{AB} \cong \overline{DE}$, $\overline{AC} \cong \overline{DF}$, and $\angle A \cong \angle D$.

   Two corresponding side lengths of the two triangles and one corresponding angle are identified as congruent.

2. Determine if this information is enough to state that all six corresponding parts of the two triangles are congruent.

   Notice that the congruent angles are included angles, meaning the angles are between the sides that are marked as congruent.

   It is given that two sides and the included angle are congruent, so the two triangles are congruent.

3. Summarize your findings.

   $\triangle ABC \cong \triangle DEF$ because of the Side-Angle-Side (SAS) Congruence Statement.
Example 3

Determine which congruence statement, if any, can be used to show that $\triangle HIJ$ and $\triangle KLM$ are congruent if $\overline{HI} \cong \overline{KL}$, $\angle H \cong \angle K$, and $\angle I \cong \angle L$.

1. Determine which components of the triangles are congruent.

One corresponding side length of the two triangles and two corresponding angles are identified as congruent.

It is often helpful to draw a diagram of the triangles with the given information to see where the congruent side is in relation to the congruent angles.

2. Determine if this information is enough to state that all six corresponding parts of the two triangles are congruent.

Notice that the congruent sides are included sides, meaning the sides are between the angles that are marked as congruent.

It is given that the two angles and the included side are equivalent, so the two triangles are congruent.

3. Summarize your findings.

$\triangle HIJ \cong \triangle KLM$ because of the Angle-Side-Angle (ASA) Congruence Statement.
Example 4

Determine which congruence statement, if any, can be used to show that \( \triangle PQR \) and \( \triangle STU \) are congruent if \( PQ \cong ST \), \( PR \cong SU \), and \( \angle Q \cong \angle T \).

1. Determine which components of the triangles are equivalent.
   
   Two corresponding side lengths of the two triangles and one corresponding angle are identified as congruent.

   Draw a diagram of the triangles with the given information to see where the congruent sides are in relation to the congruent angle.
2. Determine if this information is enough to state that all six corresponding parts of the two triangles are congruent.

Notice that the congruent angles are not included angles, meaning the angles are not between the sides that are marked as congruent.

There is no congruence statement that allows us to state that the two triangles are congruent based on the given information.

3. Summarize your findings.

It cannot be determined whether $\triangle PQR$ and $\triangle STU$ are congruent.
Example 5

Determine which congruence statement, if any, can be used to show that $\triangle ABD$ and $\triangle FEC$ are congruent.

1. Determine which components of the triangles are congruent.

Notice that the triangles overlap.
If you have trouble seeing the two triangles, redraw each triangle.

According to the diagram, $\overline{AB} \cong \overline{FE}$, $\angle A \cong \angle F$, and $\angle B \cong \angle E$.
One corresponding side length of the two triangles and two corresponding angles are identified as congruent.
2. Determine if this information is enough to state that all six corresponding parts of the two triangles are congruent.

Notice that the congruent sides are included sides, meaning the sides are between the angles that are marked as congruent.

It is given that two angles and the included side are equivalent, so the two triangles are congruent.

3. Summarize your findings.

$\triangle ABD \cong \triangle FEC$ because of the Angle-Side-Angle (ASA) Congruence Statement.
Problem-Based Task 1.5.2: Stained Glass Pattern, Part II

Mary is making a new stained glass art piece, as shown in the diagram below. As she studied the pattern, she recognized two rhombuses, each with a diagonal. She concluded that all the triangles in the pattern are congruent. Can Mary confidently state that all the triangles are congruent without using measuring tools? Use what you know about congruent triangles to prove or disprove her theory, and explain your reasoning. Remember that rhombuses have opposite angles that are congruent. Also, the diagonal of a rhombus bisects opposite angles.
Problem-Based Task 1.5.2: Stained Glass Pattern, Part II

Coaching

a. Identify the two rhombuses in the new pattern.

b. In general, what are the properties of the sides of rhombuses?

c. In general, what are the properties of the angles of rhombuses?

d. What segment is shown to be the diagonal of $\square KLDM$?

e. The diagonal of a rhombus bisects opposite angles. Which angles are bisected by $\overline{KD}$?

f. Name the angles created by $\overline{KD}$ and determine if they are congruent to one another.

g. Is it possible to determine if $\triangle DLK \cong \triangle DMK$?

h. What segment is shown to be the diagonal of $\square KMHJ$?

i. The diagonal of a rhombus bisects opposite angles. Which angles are bisected by $\overline{KH}$?

j. Name the angles created by $\overline{KH}$ and determine if they are congruent to one another.

k. Is it possible to determine if $\triangle HMK \cong \triangle HJK$?

l. Mark congruent sides and angles on the diagram.

m. Is it possible to determine if $\triangle DLK \cong \triangle HMK$?

n. Is it possible to determine if $\triangle DLK \cong \triangle HJK$?

o. Can Mary confidently state that all triangles are congruent without using measuring tools?
Problem-Based Task 1.5.2: Stained Glass Pattern, Part II

Coaching Sample Responses

a. Identify the two rhombuses in the new pattern.
   One rhombus is \( \text{\(\Box KLDM\)} \) and the second rhombus is \( \text{\(\Box KMHJ\)} \).

b. In general, what are the properties of the sides of rhombuses?
   All sides of a rhombus are congruent.

c. In general, what are the properties of the angles of rhombuses?
   Opposite angles of a rhombus are congruent.

d. What segment is shown to be the diagonal of \( \text{\(\Box KLDM\)} \)?
   \( \overline{KD} \) is the diagonal of \( \text{\(\Box KLDM\)} \).

e. The diagonal of a rhombus bisects opposite angles. Which angles are bisected by \( \overline{KD} \)?
   \( \overline{KD} \) bisects \( \angle LDM \) and \( \angle MKL \).

f. Name the angles created by \( \overline{KD} \) and determine if they are congruent to one another.
   The angles created by \( \overline{KD} \) are \( \angle LDK \), \( \angle MDK \), \( \angle LKD \), and \( \angle MKD \).
   \[ \angle LDK \cong \angle MDK \cong \angle LKD \cong \angle MKD \]
   Opposite angles of the rhombus are congruent, and the diagonal bisects them (cuts them in half). Therefore, each angle created by \( \overline{KD} \) is of the same measure. They are congruent.

g. Is it possible to determine if \( \triangle DLK \cong \triangle DMK \)?
   Yes, there is enough information to determine that \( \triangle DLK \cong \triangle DMK \).
   \[ \overline{KL} \cong \overline{LD} \cong \overline{DM} \cong \overline{MK} \]
   \[ \angle LDK \cong \angle MDK \cong \angle LKD \cong \angle MKD \]
   \[ \overline{KD} \cong \overline{KD} \]
   It follows that \( \triangle DLK \cong \triangle DMK \) because of side-angle-side (SAS) or side-side-side (SSS).

h. What segment is shown to be the diagonal of \( \text{\(\Box KMHJ\)} \)?
   \( \overline{KH} \) is the diagonal of \( \text{\(\Box KMHJ\)} \).

i. The diagonal of a rhombus bisects opposite angles. Which angles are bisected by \( \overline{KH} \)?
   \( \overline{KH} \) bisects \( \angle MHJ \) and \( \angle JKM \).
j. Name the angles created by $\overline{KH}$ and determine if they are congruent to one another.

The angles created by $\overline{KH}$ are $\angle MHK$, $\angle JHK$, $\angle MKH$, and $\angle JKH$.

$$\angle MHK \equiv \angle JHK \equiv \angle MKH \equiv \angle JKH$$

Opposite angles of the rhombus are congruent, and the diagonal bisects them, or cuts them in half. Therefore, each angle created by $\overline{KH}$ is of the same measure. They are congruent.

k. Is it possible to determine if $\triangle HMK \cong \triangle HJK$?

There is enough information to determine that $\triangle HMK \cong \triangle HJK$.

$$\overline{JH} \equiv \overline{HM} \equiv \overline{MK} \equiv \overline{KJ}$$

$$\angle MHK \equiv \angle JHK \equiv \angle MKH \equiv \angle JKH$$

$$\overline{KH} \equiv \overline{KH}$$

It follows that $\triangle HMK \cong \triangle HJK$ because of side-angle-side (SAS) or side-side-side (SSS).

l. Mark congruent sides and angles on the diagram.

m. Is it possible to determine if $\triangle DLK \cong \triangle HMK$?

It is not possible to make this determination because there is not enough information given.

The triangles only have two sets of corresponding congruent sides. It is not known whether any angles are congruent, or if the remaining two corresponding sides are congruent.

There is no congruence statement that allows us to state that the two triangles are congruent based on the given information.
n. Is it possible to determine if $\triangle DLK \cong \triangle HJK$?

It is not possible to make this determination because there is not enough information given.

The triangles only have two sets of corresponding congruent sides. It is not known whether any angles are congruent, or if the remaining two corresponding sides are congruent.

There is no congruence statement that allows us to state that the two triangles are congruent based on the given information.

o. Can Mary confidently state that all triangles are congruent without using measuring tools?

Mary cannot confidently state that all triangles are congruent without using measuring tools because not enough information is known.

**Recommended Closure Activity**

Select one or more of the essential questions for a class discussion or as a journal entry prompt.
For each diagram, determine which congruence statement can be used to show that the triangles are congruent. If it is not possible to prove triangle congruence, explain why not.

1.

2.

Practice 1.5.2: Explaining ASA, SAS, and SSS

For each diagram, determine which congruence statement can be used to show that the triangles are congruent. If it is not possible to prove triangle congruence, explain why not.
3. Based on the information in the diagram, is \( \triangle ABD \) congruent to \( \triangle CDB \)?

\[
\begin{array}{c}
A \quad B \\
\quad \\
D \quad C
\end{array}
\]

Use the given information to determine which congruence statement can be used to show that the triangles are congruent. If it is not possible to prove triangle congruence, explain why not.

4. \( \triangle STU \) and \( \triangle VWX \): \( \angle S \cong \angle V \), \( \angle T \cong \angle W \), and \( ST \cong VW \)

5. \( \triangle MNO \) and \( \triangle PQR \): \( \angle O \cong \angle R \), \( MO \cong PR \), and \( NO \cong QR \)

6. \( \triangle GHI \) and \( \triangle JKL \): \( \angle G \cong \angle H \), \( HI \cong KL \), \( \angle J \cong \angle K \)

continued
7. Jessalyn found two vintage road signs at a thrift store. She is re-decorating her room and congruency is important for her decor. Based on the information about each sign shown in the diagram below, determine if the triangles are congruent. If so, name the congruent triangles and identify the congruence statement used.

8. Isaac needs two congruent sails for his sailboat. The boat supply shop has only two sails in stock. Based on the information about each sail, determine if the sails are congruent. If so, name the congruent triangles and identify the congruence statement used.
The diagram below represents a plot of land in the town of Willow Woods. Use this diagram to solve problems 9 and 10.

9. The Kim family owns the plot of land marked by \( \triangle IKD \) and the Reed family owns the plot of land marked \( \triangle KDE \). Are the plots of land congruent? Explain your reasoning.

10. The Larsen family owns the plot of land marked by \( \triangle FGL \) and the Rodriguez family owns the plot of land marked by \( \triangle CDJ \). Are the two plots of land congruent? Explain your reasoning.
UNIT 1 • SIMILARITY, CONGRUENCE, AND PROOFS

Lesson 6: Defining and Applying Similarity

Common Core Georgia Performance Standards
MCC9–12.G.SRT.2
MCC9–12.G.SRT.3

Essential Questions
1. What does it mean for two triangles to be similar?
2. How can you prove that two triangles are similar?
3. How can you use similar triangles to solve problems?

WORDS TO KNOW

Angle-Angle (AA) Similarity Statement: If two angles of one triangle are congruent to two angles of another triangle, then the triangles are similar.

proportional: having a constant ratio to another quantity

ratio of similitude: a ratio of corresponding sides; also known as the scale factor

similar: two figures that are the same shape but not necessarily the same size; the symbol for representing similarity between figures is ~

similarity transformation: a rigid motion followed by a dilation; a transformation that results in the position and size of a figure changing, but not the shape

Recommended Resources

  http://www.walch.com/rr/00021
  This site provides examples and illustrations of similarity criteria.

- Math Open Reference. “Similar Triangles.”
  http://www.walch.com/rr/00022
  This site includes a summary of similarity as well as links to tests of similarity.

  http://www.walch.com/rr/00023
  This site provides a summary of similar figures and how to calculate lengths of sides of similar figures.
Lesson 1.6.1: Defining Similarity

Warm-Up 1.6.1

A poster of the tallest buildings in the world hangs in a hallway. The scale on the poster is 0.5 inches = 45 feet.

1. The height of the tallest building in the world, Burj Khalifa in Dubai, is about 30.25 inches on the poster. What is the actual height of Burj Khalifa?

2. The height of the tallest building in the United States, Willis Tower (formerly known as Sears Tower), is about 16 inches on the poster. What is the actual height of Willis Tower?

3. How much taller is Burj Khalifa than Willis Tower?
Lesson 1.6.1: Defining Similarity

Common Core Georgia Performance Standard

MCC9–12.G.SRT.2

Warm-Up 1.6.1 Debrief

A poster of the tallest buildings in the world hangs in a hallway. The scale on the poster is 0.5 inches = 45 feet.

1. The height of the tallest building in the world, Burj Khalifa in Dubai, is about 30.25 inches on the poster. What is the actual height of Burj Khalifa?

Set up a proportion. Let \( h \) represent the actual height of the building.

\[
\frac{0.5}{45} = \frac{30.25}{h}
\]

Proportion

\[0.5h = 45(30.25)\]

Simplify through cross-multiplication.

\[0.5h = 1361.25\]

Solve for \( h \).

\[h = 2722.5\]

The actual height of Burj Khalifa is approximately 2,722.5 feet.

2. The height of the tallest building in the United States, Willis Tower (formerly known as Sears Tower), is about 16 inches on the poster. What is the actual height of Willis Tower?

Create a proportion to find the actual height of the building, \( h \).

\[
\frac{0.5}{45} = \frac{16}{h}
\]

Proportion

\[0.5h = 45(16)\]

Simplify through cross-multiplication.

\[0.5h = 720\]

Solve for \( h \).

\[h = 1440\]

The actual height of Willis Tower is approximately 1,440 feet.
3. How much taller is Burj Khalifa than Willis Tower?

Find the difference of the calculated heights of the two buildings.

\[ 2722.5 - 1440 = 1282.5 \]

Burj Khalifa is approximately 1,282.5 feet taller than Willis Tower.

**Connection to the Lesson**

- Students will use ratios and proportions to determine the ratio of similitude and calculate missing lengths of similar triangles.
Prerequisite Skills
This lesson requires the use of the following skills:

- creating ratios
- solving proportions
- identifying congruent triangles
- calculating the lengths of triangle sides using the distance formula
- recognizing transformations performed as a combination of translations, reflections, rotations, and/or dilations

Introduction
Congruent triangles have corresponding parts with angle measures that are the same and side lengths that are the same. If two triangles are congruent, they are also similar. Similar triangles have the same shape, but may be different in size. It is possible for two triangles to be similar but not congruent. Just like with determining congruency, it is possible to determine similarity based on the angle measures and lengths of the sides of the triangles.

Key Concepts

- To determine whether two triangles are similar, observe the angle measures and the side lengths of the triangles.
- When a triangle is transformed by a similarity transformation (a rigid motion [reflection, translation, or rotation] followed by a dilation), the result is a triangle with a different position and size, but the same shape.
- If two triangles are similar, then their corresponding angles are congruent and the measures of their corresponding sides are proportional, or have a constant ratio.
- The ratio of corresponding sides is known as the ratio of similitude.
- The scale factor of the dilation is equal to the ratio of similitude.
- Similar triangles with a scale factor of 1 are congruent triangles.
- Like with congruent triangles, corresponding angles and sides can be determined by the order of the letters.
- If $\triangle ABC$ is similar to $\triangle DEF$, the vertices of the two triangles correspond in the same order as they are named.
• The symbol → shows that parts are corresponding.
  \[ \angle A \rightarrow \angle D \text{; they are equivalent.} \]
  \[ \angle B \rightarrow \angle E \text{; they are equivalent.} \]
  \[ \angle C \rightarrow \angle F \text{; they are equivalent.} \]

• The corresponding angles are used to name the corresponding sides.
  \[ \overline{AB} \rightarrow \overline{DE} \]
  \[ \overline{BC} \rightarrow \overline{EF} \]
  \[ \overline{AC} \rightarrow \overline{DF} \]

  \[
  \frac{AB}{DE} = \frac{BC}{EF} = \frac{AC}{DF}
  \]

• Observe the diagrams of \( \triangle ABC \) and \( \triangle DEF \).

• The symbol for similarity (\( \sim \)) is used to show that figures are similar.
  \[ \triangle ABC \sim \triangle DEF \]

\[ \angle A \cong \angle D \]
\[ \angle B \cong \angle E \]
\[ \angle C \cong \angle F \]

\[
\frac{AB}{DE} = \frac{BC}{EF} = \frac{AC}{DF}
\]

**Common Errors/Misconceptions**

- incorrectly identifying corresponding parts of triangles
- assuming corresponding parts indicate congruent parts
- assuming alphabetical order indicates congruence
- changing the order of named triangles, causing parts to be incorrectly interpreted as congruent
Guided Practice 1.6.1

Example 1

Use the definition of similarity in terms of similarity transformations to determine whether the two figures are similar. Explain your answer.

1. Examine the orientation of the triangles.

<table>
<thead>
<tr>
<th>(\triangle ABC)</th>
<th>(\triangle XYZ)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Side</td>
<td>Orientation</td>
</tr>
<tr>
<td>(\overline{AB})</td>
<td>Top left side of triangle</td>
</tr>
<tr>
<td>(\overline{AC})</td>
<td>Bottom left side of triangle</td>
</tr>
<tr>
<td>(\overline{BC})</td>
<td>Right side of triangle</td>
</tr>
</tbody>
</table>

The orientation of the triangles has remained the same, indicating translation, dilation, stretch, or compression.
2. Determine whether a dilation has taken place by calculating the scale factor.

First, identify the vertices of each triangle.

\[\text{\(A\)} (-4, 1), \text{\(B\)} (1, 4), \text{\(C\)} (2, -2)\]

\[\text{\(X\)} (-8, 2), \text{\(Y\)} (2, 8), \text{\(Z\)} (4, -4)\]

Then, find the length of each side of \(\triangle ABC\) and \(\triangle XYZ\) using the distance formula, \(d = \sqrt{(x_2-x_1)^2 + (y_2-y_1)^2}\).

Calculate the distance of \(\overline{AB}\).

\[
\begin{align*}
\text{Distance formula} \\
d &= \sqrt{(x_2-x_1)^2 + (y_2-y_1)^2} \\
d &= \sqrt{((1)-(-4))^2 + ((4)-(1))^2} \\
d &= \sqrt{(5)^2 + (3)^2} \\
d &= \sqrt{25 + 9} \\
d &= \sqrt{34}
\end{align*}
\]

The distance of \(\overline{AB}\) is \(\sqrt{34}\) units.

Calculate the distance of \(\overline{BC}\).

\[
\begin{align*}
\text{Distance formula} \\
d &= \sqrt{(x_2-x_1)^2 + (y_2-y_1)^2} \\
d &= \sqrt{((2)-(1))^2 + ((-2)-(4))^2} \\
d &= \sqrt{(1)^2 + (-6)^2} \\
d &= \sqrt{1 + 36} \\
d &= \sqrt{37}
\end{align*}
\]

The distance of \(\overline{BC}\) is \(\sqrt{37}\) units. (continued)
Calculate the distance of \( \overline{AC} \).

\[
d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}
\]

Distance formula

\[
d = \sqrt{[(2) - (-4)]^2 + [(-2) - (1)]^2}
\]

Substitute \((-4, 1)\) and \((2, -2)\) for \((x_1, y_1)\) and \((x_2, y_2)\).

\[
d = \sqrt{(6)^2 + (-3)^2}
\]

Simplify.

\[
d = \sqrt{36 + 9}
\]

\[
d = \sqrt{45} = \sqrt{9 \cdot 5} = 3\sqrt{5}
\]

The distance of \( \overline{AC} \) is \( 3\sqrt{5} \) units.

Calculate the distance of \( \overline{XY} \).

\[
d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}
\]

Distance formula

\[
d = \sqrt{[(2) - (-8)]^2 + [(8) - (2)]^2}
\]

Substitute \((-8, 2)\) and \((2, 8)\) for \((x_1, y_1)\) and \((x_2, y_2)\).

\[
d = \sqrt{(10)^2 + (6)^2}
\]

Simplify.

\[
d = \sqrt{100 + 36}
\]

\[
d = \sqrt{136} = \sqrt{4 \cdot 34} = 2\sqrt{34}
\]

The distance of \( \overline{XY} \) is \( 2\sqrt{34} \) units.
Calculate the distance of $YZ$.

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$  
Distance formula

$$d = \sqrt{(4 - (2))^2 + ((-4) - (8))^2}$$  
Substitute $(2, 8)$ and $(4, -4)$ for $(x_1, y_1)$ and $(x_2, y_2)$.

$$d = \sqrt{(2)^2 + (-12)^2}$$  
Simplify.

$$d = \sqrt{4 + 144}$$

$$d = \sqrt{148} = \sqrt{4 \cdot 37} = 2\sqrt{37}$$

The distance of $YZ$ is $2\sqrt{37}$ units.

Calculate the distance of $XZ$.

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$  
Distance formula

$$d = \sqrt{((-8) - (4))^2 + ((-4) - (2))^2}$$  
Substitute $(-8, 2)$ and $(4, -4)$ for $(x_1, y_1)$ and $(x_2, y_2)$.

$$d = \sqrt{(12)^2 + (-6)^2}$$  
Simplify.

$$d = \sqrt{144 + 36}$$

$$d = \sqrt{180} = \sqrt{36 \cdot 5} = 6\sqrt{5}$$

The distance of $XZ$ is $6\sqrt{5}$ units.
3. Calculate the scale factor of the changes in the side lengths.
   Divide the side lengths of \( \triangle XYZ \) by the side lengths of \( \triangle ABC \).

   \[
   \frac{XY}{AB} = \frac{2\sqrt{34}}{\sqrt{34}} = 2
   \]

   \[
   \frac{YZ}{BC} = \frac{2\sqrt{37}}{\sqrt{37}} = 2
   \]

   \[
   \frac{XZ}{AC} = \frac{6\sqrt{5}}{3\sqrt{5}} = 2
   \]

   The scale factor is constant between each pair of corresponding sides.

4. Determine if another transformation has taken place.
   Multiply the coordinate of each vertex of the preimage by the scale factor, \( k \).

   \[
   D_k(x, y) = (kx, ky)
   \]

   Dilation by a scale factor of \( k \)

   \[
   D_2(-4, 1) = [2(-4), 2(1)] = (-8, 2)
   \]

   \[
   D_2(1, 4) = [2(1), 2(4)] = (2, 8)
   \]

   \[
   D_2(2, -2) = [2(2), 2(-2)] = (4, -4)
   \]

   You can map \( \triangle ABC \) onto \( \triangle XYZ \) by the dilation with a scale factor of 2.

5. State your conclusion.
   A dilation is a similarity transformation; therefore, \( \triangle ABC \) and \( \triangle XYZ \) are similar. The ratio of similitude is 2.
Example 2
Use the definition of similarity in terms of similarity transformations to determine whether the two figures are similar. Explain your answer.

1. Examine the angle measures of the triangles.
   Use a protractor or construction methods to determine if corresponding angles are congruent.
   None of the angles of $\triangle HJK$ are congruent to the angles of $\triangle NPQ$.

2. Summarize your findings.
   Similarity transformations preserve angle measure.
   The angles of $\triangle HJK$ and $\triangle NPQ$ are not congruent.
   There are no sequences of transformations that will map $\triangle HJK$ onto $\triangle NPQ$.
   $\triangle HJK$ and $\triangle NPQ$ are not similar triangles.
Example 3

A dilation of $\triangle TUV$ centered at point $P$ with a scale factor of 2 is then reflected over the line $l$. Determine if $\triangle TUV$ is similar to $\triangle DEF$. If possible, find the unknown angle measures and lengths in $\triangle DEF$.

1. Determine if $\triangle TUV$ and $\triangle DEF$ are similar.

   The transformations performed on $\triangle TUV$ are dilation and reflection.

   The sequence of dilating and reflecting a figure is a similarity transformation; therefore, $\triangle TUV \sim \triangle DEF$.
2. Identify the angle measures of $\triangle DEF$.

Corresponding angles of similar triangles are congruent.

$\angle T \cong \angle D$

$\angle U \cong \angle E$

$\angle V \cong \angle F$

Since $\angle U$ is marked as a right angle, $\angle E$ must also be a right angle.

3. Identify the known lengths of $\triangle DEF$.

Corresponding sides of similar triangles are proportional.

The ratio of similitude is equal to the scale factor used in the dilation of the figure.

$$\frac{DE}{TU} = \frac{EF}{UV} = \frac{DF}{TV} = 2$$

Since the length of $TU$ is 3 units, the length of the corresponding side $DE$ can be found using the ratio of similitude.

$$\frac{DE}{3} = 2$$

$DE = 6$

$\overline{DE}$ is 6 units long.

Since the length of $TV$ is 5 units, the length of the corresponding side $DF$ can be found.

$$\frac{DF}{5} = 2$$

$DF = 10$

$\overline{DF}$ is 10 units long.
Problem-Based Task 1.6.1: Video Game Transformations

The creators of a new video game are in the early design stages and are using the right triangle $ABC$ on a coordinate plane to represent the movement of the character in the actual game setting. The coordinates of the points are $A(-6, -9)$, $B(-3, -9)$, and $C(-3, -6)$. After a series of similarity transformations, the locations of the end points of the hypotenuse of the new image are $A'(6, -4)$ and $C'(4, -2)$. Is it possible to determine the location of point $B'$?
Problem-Based Task 1.6.1: Video Game Transformations

Coaching

a. What is the length of hypotenuse $\overline{AC}$?

b. What is the length of $\overline{A'C'}$?

c. Is the length of $\overline{A'C'}$ shorter or longer than the length of $\overline{AC}$?

d. By what scale factor was the length of $\overline{AC}$ changed?

e. If a series of similarity transformations were carried out, what would you know about the lengths of the remaining sides of the image compared to the lengths of the preimage?

f. What is the length of $\overline{AB}$?

g. What is the length of $\overline{A'B'}$?

h. What is the length of $\overline{BC}$?

i. What is the length of $\overline{B'C'}$?

j. What are the possible locations of point $B'$?
Problem-Based Task 1.6.1: Video Game Transformations

Coaching Sample Responses

a. What is the length of hypotenuse \(AC\)?

Use the distance formula to calculate the length of \(AC\).

\[
d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}
\]

Distance formula

\[
d = \sqrt{[(-3) - (-6)]^2 + [(-6) - (-9)]^2}
\]

Substitute \((-6, -9)\) and \((-3, -6)\) for \((x_1, y_1)\) and \((x_2, y_2)\).

\[
d = \sqrt{3^2 + 3^2}
\]

Simplify.

\[
d = \sqrt{18} = \sqrt{9 \cdot 2} = 3\sqrt{2}
\]

The length of \(AC\) is \(3\sqrt{2}\) units.

b. What is the length of \(A'C'\)?

Use the distance formula to calculate the length of \(A'C'\).

\[
d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}
\]

Distance formula

\[
d = \sqrt{[(4) - (-6)]^2 + [(-2) - (-4)]^2}
\]

Substitute \((6, -4)\) and \((4, -2)\) for \((x_1, y_1)\) and \((x_2, y_2)\).

\[
d = \sqrt{(-2)^2 + (2)^2}
\]

Simplify.

\[
d = \sqrt{8} = \sqrt{4 \cdot 2} = 2\sqrt{2}
\]

The length of \(A'C'\) is \(2\sqrt{2}\) units.

c. Is the length of \(A'C'\) shorter or longer than the length of \(AC\)?

The length of \(AC\) is \(3\sqrt{2}\).

The length of \(A'C'\) is \(2\sqrt{2}\) units.

The length of \(A'C'\) is shorter than the length of \(AC\).
d. By what scale factor was the length of $\overline{AC}$ changed?

Divide the length of $\overline{A'C'}$ by the length of $\overline{AC}$.

$$\frac{A'C'}{AC} = \frac{2\sqrt{2}}{3\sqrt{2}} = \frac{2}{3}$$

$\overline{A'C'}$ is $\frac{2}{3}$ the length of $\overline{AC}$.

e. If a series of similarity transformations were carried out, what would you know about the lengths of the remaining sides of the image compared to the lengths of the preimage?

The lengths of the sides of the image are all proportional to the preimage.

Each side of $\triangle A'B'C'$ is $\frac{2}{3}$ the length of the sides of $\triangle ABC$.

f. What is the length of $\overline{AB}$?

Use the distance formula to calculate the length of $\overline{AB}$.

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

Distance formula

$$d = \sqrt{[(-3) - (-6)]^2 + [(-9) - (-9)]^2}$$

Substitute $(-6, -9)$ and $(-3, -9)$ for $(x_1, y_1)$ and $(x_2, y_2)$.

$$d = \sqrt{3^2 + 0^2}$$

Simplify.

$$d = \sqrt{9} = 3$$

The length of $\overline{AB}$ is 3 units.

g. What is the length of $\overline{A'B'}$?

Use the scale factor of $\frac{2}{3}$ to find the length of $\overline{A'B'}$.

$$\frac{2}{3} \cdot 3 = 2$$

The length of $\overline{A'B'}$ is 2 units.
h. What is the length of $BC$?

Use the distance formula to calculate the length of $BC$.

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

Distance formula

$$d = \sqrt{((-3) - (-3))^2 + ((-6) - (-9))^2}$$

Substitute $(-3, -9)$ and $(-3, -6)$ for $(x_1, y_1)$ and $(x_2, y_2)$.

$$d = \sqrt{(0)^2 + (3)^2}$$

Simplify.

$$d = \sqrt{9} = 3$$

The length of $BC$ is 3 units.

i. What is the length of $B'C'$?

Use the scale factor of $\frac{2}{3}$ to find the length of $B'C'$.

$$\frac{2}{3} \times 3 = 2$$

The length of $B'C'$ is 2 units.

j. What are the possible locations of point $B'$?

There are two possible locations for point $B'$.

One location for point $B'$ is 2 units to the right of $C'$ and 2 units up from $A'$.

This location is at $(4, -4)$.

The second location is 2 units down from $C'$ and 2 units to the left of $A'$.

This location is at $(6, -2)$.

**Recommended Closure Activity**

Select one or more of the essential questions for a class discussion or as a journal entry prompt.
Practice 1.6.1: Defining Similarity

Find all the angle measures and side lengths for each triangle of the given similar pairs.

1. $\triangle ABC \sim \triangle DEF$

2. $\triangle JKL \sim \triangle MNP$
Determine if the two given triangles are similar. Use the definition of similarity in terms of similarity transformations to explain your answer.

3.

4.

continued
5. \[ \begin{array}{c}
\text{Diagram showing points A, B, C, D, E, and F on a coordinate plane.} \\
\end{array} \]

6. \[ \begin{array}{c}
\text{Diagram showing points A, B, C, D, E, and F on a coordinate plane.} \\
\end{array} \]

(continued)
Lesson 6: Defining and Applying Similarity

7.

8.

continued
9. [Diagram of triangles A, B, C, D, E, F with coordinates labeled]

10. [Diagram of triangles A, B, C, D, E, F with coordinates labeled]
The Statue of Liberty, a gift to the United States from France in 1886, stands 93 meters tall. A replica of the famous statue is 7.75 meters.

1. What is the scale factor comparing the height of the replica to the actual height of the Statue of Liberty?

2. The height of the base of the Statue of Liberty is approximately 46.5 meters. What is the height of the base of the Statue of Liberty replica?
Lesson 1.6.2: Applying Similarity Using the Angle-Angle (AA) Criterion

Common Core Georgia Performance Standard

MCC9–12.G.SRT.3

Warm-Up 1.6.2 Debrief

The Statue of Liberty, a gift to the United States from France in 1886, stands 93 meters tall. A replica of the famous statue is 7.75 meters.

1. What is the scale factor comparing the height of the replica to the actual height of the Statue of Liberty?

Create a ratio comparing the height of the replica to the actual height of the statue.

\[
\frac{\text{height of replica}}{\text{actual height}} = \frac{7.75 \text{ meters}}{93 \text{ meters}} = \frac{1}{12}
\]

The ratio comparing the two heights is \(\frac{1}{12}\).

The scale factor is \(\frac{1}{12}\).

2. The height of the base of the Statue of Liberty is approximately 46.5 meters. What is the height of the base of the Statue of Liberty replica?

Write a proportion to find the height of the base of the replica.

\[
\frac{\text{replica base height}}{\text{actual base height}} = \frac{1}{12}
\]

\[
\frac{x}{46.5 \text{ meters}} = \frac{1}{12}
\]

\[12x = 46.5\]

\[x = 3.875\]

The height of the base of the Statue of Liberty replica is 3.875 meters.

Connection to the Lesson

- Students will continue to use ratios and proportions to determine the ratio of similitude and calculate missing lengths of similar triangles.
**Prerequisite Skills**
This lesson requires the use of the following skills:
- understanding that the sum of the measures of the angles in a triangle is 180°
- identifying both corresponding and congruent parts of triangles

**Introduction**
When a series of similarity transformations are performed on a triangle, the result is a similar triangle. When triangles are similar, the corresponding angles are congruent and the corresponding sides are of the same proportion. It is possible to determine if triangles are similar by measuring and comparing each angle and side, but this can take time. There exists a set of similarity statements, similar to the congruence statements, that let us determine with less information whether triangles are similar.

**Key Concepts**
- The **Angle-Angle (AA) Similarity Statement** is one statement that allows us to prove triangles are similar.
- The AA Similarity Statement allows that if two angles of one triangle are congruent to two angles of another triangle, then the triangles are similar.

\[ \triangle ABC \sim \triangle XYZ \]

- Notice that it is not necessary to show that the third pair of angles is congruent because the sum of the angles must equal 180°.
- Similar triangles have corresponding sides that are proportional.
- The Angle-Angle Similarity Statement can be used to solve various problems, including those that involve indirect measurement, such as using shadows to find the height of tall structures.
Common Errors/Misconceptions

- misidentifying congruent parts because of the orientation of the triangles
- misreading similarity statements as congruent statements
- incorrectly creating proportions between corresponding sides
Guided Practice 1.6.2

Example 1

Explain why the triangles are similar and write a similarity statement.

1. Identify the given information.
   According to the diagram, \( \angle A \cong \angle D \) and \( \angle B \cong \angle E \).

2. State your conclusion.
   \( \triangle ABC \sim \triangle DEC \) by the Angle-Angle (AA) Similarity Statement.

Example 2

Explain why \( \triangle ABC \sim \triangle DEF \), and then find the length of \( DF \).

1. Show that the triangles are similar.
   According to the diagram, \( \angle A \cong \angle D \) and \( \angle C \cong \angle F \).
   \( \triangle ABC \sim \triangle DEF \) by the Angle-Angle (AA) Similarity Statement.
2. Find the length of $DF$.

Corresponding sides of similar triangles are proportional.

Create and solve a proportion to find the length of $DF$.

\[
\frac{AB}{DE} = \frac{AC}{DF} \quad \text{Corresponding sides are proportional.}
\]

\[
\frac{3.4}{2.72} = \frac{6.9}{x} \quad \text{Substitute known values. Let } x \text{ represent the length of } DF.
\]

\[
(2.72)(6.9) = (3.4)(x) \quad \text{Solve for } x.
\]

\[
18.768 = 3.4x
\]

\[
x = 5.52
\]

The length of $DF$ is 5.52 units.

Example 3

Identify the similar triangles. Find $x$ and the measures of the indicated sides.

1. Show that the triangles are similar.

   According to the diagram, $\angle A \cong \angle H$ and $\angle C \cong \angle J$.

   $\triangle ABC \sim \triangle HGJ$ by the Angle-Angle (AA) Similarity Statement.
2. Use the definition of similar triangles to find the value of \( x \).

Corresponding sides of similar triangles are proportional.

Create and solve a proportion to find the value of \( x \).

\[
\frac{AB}{DE} = \frac{AC}{DF}
\]

Corresponding sides are proportional.

\[
\frac{x + 1}{3} = \frac{x + 5}{6}
\]

Substitute known values.

\[
(x + 1)(6) = (3)(x + 5)
\]

Solve for \( x \).

\[
6x + 6 = 3x + 15
\]

\[
3x = 9
\]

\[
x = 3
\]

3. Find the unknown side lengths.

Use the value of \( x \) to find the unknown lengths of the triangles.

\[
AB = x + 1
\]

\[
= 3 + 1
\]

\[
= 4
\]

\[
AC = x + 5
\]

\[
= 3 + 5
\]

\[
= 8
\]

The length of \( AB \) is 4 units.

The length of \( AC \) is 8 units.
Example 4

Suppose a person 5 feet 10 inches tall casts a shadow that is 3 feet 6 inches long. At the same time of day, a flagpole casts a shadow that is 12 feet long. To the nearest foot, how tall is the flagpole?

1. Identify the known information.
   - The height of a person and the length of the shadow cast create a right angle.
   - The height of the flagpole and the length of the shadow cast create a second right angle.
   - You can use this information to create two triangles.
   - Draw a picture to help understand the information.

2. Determine if the triangles are similar.
   - Two pairs of angles are congruent.
   - According to the Angle-Angle (AA) Similarity Statement, the triangles are similar.
   - Corresponding sides of similar triangles are proportional.
3. Find the height of the flagpole.

Create and solve a proportion to find the height of the flagpole.

\[
\frac{5\frac{10}{12}}{x} = \frac{3\frac{6}{12}}{12}
\]

Corresponding sides are proportional.

Let \(x\) represent the height of the flagpole.

\[
\frac{5\frac{5}{6}}{x} = \frac{3\frac{1}{2}}{12}
\]

Simplify.

\[
(5\frac{5}{6})(12) = (3\frac{1}{2})(x)
\]

Solve for \(x\).

\[
70 = (3\frac{1}{2})(x)
\]

\[
x = 20
\]

The flagpole is 20 feet tall.
Problem-Based Task 1.6.2: True Trusses

A mono truss is a type of building support structure that is in the shape of a right triangle. Contractors often use mono trusses when building roofs for small structures such as garages and sheds. The vertical pieces of this truss form 90° angles with the horizontal pieces in order to maximize the stability. Observe the diagram of a mono truss below. Is \( \triangle ABC \) similar to \( \triangle ADE \)? Explain your reasoning. Is it possible to determine the length of \( DE \) from the given information? If so, calculate the length.
Problem-Based Task 1.6.2: True Trusses

Coaching

a. Which angles of $\triangle ABC$ are congruent to angles of $\triangle ADE$? Name them, and explain why they are congruent.

b. Is there enough information to determine if $\triangle ABC$ is similar to $\triangle ADE$?

c. If two triangles are similar, what must be true about the lengths of their corresponding sides?

d. Identify the corresponding sides of $\triangle ABC$ and $\triangle ADE$.

e. What is the ratio between the length of $\overline{AC}$ and the length of $\overline{AE}$?

f. What is the length of $\overline{DE}$?
Problem-Based Task 1.6.2: True Trusses

Coaching Sample Responses

a. Which angles of $\triangle ABC$ are congruent to angles of $\triangle ADE$? Name them, and explain why they are congruent.

$\angle BCA$ of $\triangle ABC$ is congruent to $\angle DEA$ of $\triangle ADE$ because they are both right angles.

$\angle BAC$ of $\triangle ABC$ is congruent to $\angle DAE$ of $\triangle ADE$ because they are composed of the same angle, $\angle A$.

b. Is there enough information to determine if $\triangle ABC$ is similar to $\triangle ADE$?

Yes; according to the Angle-Angle (AA) Similarity Statement, if two angles of one triangle are congruent to two angles of another triangle, then the triangles are similar.

c. If two triangles are similar, what must be true about the lengths of their corresponding sides?

Similar triangles have corresponding sides that are proportional.

d. Identify the corresponding sides of $\triangle ABC$ and $\triangle ADE$.

$\overline{AB} \rightarrow \overline{AD}$

$\overline{AC} \rightarrow \overline{AE}$

$\overline{BC} \rightarrow \overline{DE}$

e. What is the ratio between the length of $\overline{AC}$ and the length of $\overline{AE}$?

$$\frac{AC}{AE} = \frac{5.25 + 4.75 + 4}{5.25} = \frac{14}{5.25} = \frac{8}{3}$$

f. What is the length of $\overline{DE}$?

Create and solve a proportion using the ratio found in part e.

$$\frac{BC}{DE} = \frac{8}{3} = \frac{4.5}{x}$$

$$x = 1.6875$$

The length of $\overline{DE}$ is 1.6875 feet.

Recommended Closure Activity

Select one or more of the essential questions for a class discussion or as a journal entry prompt.
Practice 1.6.2: Applying Similarity Using the Angle-Angle (AA) Criterion

Decide whether each pair of triangles is similar. Explain your answer.

1.

2.

3.

continued
Identify the similar triangles. Find $x$ and the measures of the indicated sides.

4. 

5. 

6. 

continued
Use the definition of similarity to solve each problem.

7. At a certain time of day, a tree that is 12 feet tall casts a shadow that is 8 feet long. Find the length of the shadow that is created by a 10-foot-tall basketball hoop at the same time of day.

8. Sheila is standing near the Eiffel Tower in Paris, France. The shadow of the monument is 580 feet long, and Sheila’s shadow is 3 feet long. If Sheila is 5 feet 6 inches tall, how tall is the monument?
9. The support beams of truss bridges are triangles. James made a model of a truss bridge with a scale of 1 inch = 4 feet. If the height of the tallest triangle on the model is 9 inches, what is the height of the tallest triangle on the actual bridge?

10. A statue that is 25 feet tall casts a shadow that is 16 feet long. A cement post next to the statue is 4 feet tall. Find the length of the cement post’s shadow.
UNIT 1 • SIMILARITY, CONGRUENCE, AND PROOFS

Lesson 7: Proving Similarity

Common Core Georgia Performance Standards
MCC9–12.G.SRT.4
MCC9–12.G.SRT.5

Essential Questions
1. How can you prove that two triangles are similar?
2. How does a line parallel to one side of a triangle divide the second side of the triangle?
3. How can you use triangle similarity to prove the Pythagorean Theorem?
4. How can you use similar triangles to solve problems?
5. How can you use similarity and congruence to solve problems?

WORDS TO KNOW
altitude the perpendicular line from a vertex of a figure to its opposite side; height
angle bisector a ray that divides an angle into two congruent angles
converse of the Pythagorean Theorem If the sum of the squares of the measures of two sides of a triangle equals the square of the measure of the longest side, then the triangle is a right triangle.
flow proof a graphical method of presenting the logical steps used to show an argument. In a flow proof, the logical statements are written in boxes and the reason for each statement is written below the box.
paragraph proof statements written out in complete sentences in a logical order to show an argument
parallel lines lines in a plane that either do not share any points and never intersect, or share all points; written as \( \overline{AB} \parallel \overline{PQ} \)
proof a set of justified statements organized to form a convincing argument that a given statement is true
Reflexive Property of Congruent Segments a segment is congruent to itself; \( \overline{AB} \cong \overline{AB} \)
Lesson 7: Proving Similarity

**Segment Addition Postulate**  If \( B \) is between \( A \) and \( C \), then \( AB + BC = AC \). Conversely, if \( AB + BC = AC \), then \( B \) is between \( A \) and \( C \).

**Side-Angle-Side (SAS) Similarity Statement**  If the measures of two sides of a triangle are proportional to the measures of two corresponding sides of another triangle and the included angles are congruent, then the triangles are similar.

**Side-Side-Side (SSS) Similarity Statement**  If the measures of the corresponding sides of two triangles are proportional, then the triangles are similar.

**Symmetric Property of Congruent Segments**  If \( AB \cong CD \), then \( CD \cong AB \).

**theorem**  a statement that is shown to be true

**Transitive Property of Congruent Segments**  If \( AB \cong CD \), and \( CD \cong EF \), then \( AB \cong EF \).

**two-column proof**  numbered statements and corresponding reasons that show the argument in a logical order

**Recommended Resources**

- Learn Zillion. “Prove the Pythagorean Theorem.”
  

  This site offers a video explaining the Pythagorean Theorem through similar triangles as well as links to practice questions and questions to support understanding.

- Math Open Reference. “Similar Triangles.”
  

  This site includes a summary of similarity and links to tests of similarity.

  

  This site includes a summary of the Angle Bisector Theorem in addition to illustrated examples.
Lesson 1.7.1: Proving Triangle Similarity Using Side-Angle-Side (SAS) and Side-Side-Side (SSS) Similarity

Warm-Up 1.7.1

Dasha is making earrings to give her friend as a birthday gift. She’s designed a pattern of 3 triangles stacked on top of one another. Dasha is missing some of the measurements for her project. The given units in the diagram are in centimeters.

1. Are the triangles similar? Explain your answer.

2. Is it possible to determine the missing lengths, \( x \), \( y \), and \( z \)? If so, find each value.
Warm-Up 1.7.1 Debrief

Dasha is making earrings to give her friend as a birthday gift. She’s designed a pattern of 3 triangles stacked on top of one another. Dasha is missing some of the measurements for her project. The given units in the diagram are in centimeters.

1. Are the triangles similar? Explain your answer.

Two angles in each triangle are congruent, as noted by the arc marks.

According to the Angle-Angle Similarity Statement, if two angles of one triangle are congruent to two angles of another triangle, then the triangles are similar.

\[ \triangle ABC \sim \triangle DEF \sim \triangle GHJ \]

2. Is it possible to determine the missing lengths, \( x \), \( y \), and \( z \)? If so, find each value.

If two triangles are similar, then the measures of their corresponding sides are proportional, or have a constant ratio.

Create a proportion to find each of the unknown values.
Find the value of $x$.

\[
\frac{AC}{GJ} = \frac{AB}{GH} \quad \text{Create a proportion.}
\]

\[
\frac{2}{13.75} = \frac{5}{x} \quad \text{Substitute.}
\]

\[
(2)(13.75) = (5)(x) \quad \text{Find the cross products.}
\]

\[
27.5 = 5x \quad \text{Simplify.}
\]

\[
x = 5.5 \quad \text{Solve for } x.
\]

Find the value of $y$.

\[
\frac{AC}{DF} = \frac{AB}{DE} \quad \text{Create a proportion.}
\]

\[
\frac{2}{3.5} = \frac{5}{y} \quad \text{Substitute.}
\]

\[
(2)(y) = (3.5)(5) \quad \text{Find the cross products.}
\]

\[
2y = 17.5 \quad \text{Simplify.}
\]

\[
y = 8.75 \quad \text{Solve for } y.
\]

Find the value of $z$.

\[
\frac{AC}{DF} = \frac{BC}{EF} \quad \text{Create a proportion.}
\]

\[
\frac{2}{3.5} = \frac{z}{6.125} \quad \text{Substitute.}
\]

\[
(2)(6.125) = (3.5)(z) \quad \text{Find the cross products.}
\]

\[
12.25 = 3.5z \quad \text{Simplify.}
\]

\[
z = 3.5 \quad \text{Solve for } z.
\]

The missing values for $x$, $y$, and $z$ are 5.5 cm, 8.75 cm, and 3.5 cm, respectively.

**Connection to the Lesson**

- Students will continue learning about similar triangles and similarity statements. Students will apply their understanding of the Angle-Angle Similarity Statement as they learn about the Side-Angle-Side and Side-Side-Side similarity statements.
Introduction

There are many ways to show that two triangles are similar, just as there are many ways to show that two triangles are congruent. The Angle-Angle (AA) Similarity Statement is one of them. The Side-Angle-Side (SAS) and Side-Side-Side (SSS) similarity statements are two more ways to show that triangles are similar. In this lesson, we will prove that triangles are similar using the similarity statements.

Key Concepts

- The **Side-Angle-Side (SAS) Similarity Statement** asserts that if the measures of two sides of a triangle are proportional to the measures of two corresponding sides of another triangle and the included angles are congruent, then the triangles are similar.

- Similarity statements identify corresponding parts just like congruence statements do.

\[ \triangle ABC \sim \triangle DEF \]

\[ \angle B \cong \angle E \]

\[ DE = (x)AB \]

\[ EF = (x)BC \]
• The **Side-Side-Side (SSS) Similarity Statement** asserts that if the measures of the corresponding sides of two triangles are proportional, then the triangles are similar.

\[ \triangle ABC \sim \triangle DEF \]

\[ DE = (x)AB \]
\[ EF = (x)BC \]
\[ DF = (x)AC \]

• It is important to note that while both similarity and congruence statements include an SSS and an SAS statement, the statements do not mean the same thing.

• Similar triangles have corresponding sides that are proportional, whereas congruent triangles have corresponding sides that are of the same length.

• Like with the Angle-Angle Similarity Statement, both the Side-Angle-Side and the Side-Side-Side similarity statements can be used to solve various problems.

• The ability to prove that triangles are similar is essential to solving many problems.

• A **proof** is a set of justified statements organized to form a convincing argument that a given statement is true.

• Definitions, algebraic properties, and previously proven statements can be used to prove a given statement.

• There are several types of proofs, such as paragraph proofs, two-column proofs, and flow diagrams.
• Every good proof includes the following:
  • a statement of what is to be proven
  • a list of the given information
  • if possible, a diagram including the given information
  • step-by-step statements that support your reasoning

**Common Errors/Misconceptions**

• misidentifying congruent parts because of the orientation of the triangles
• misreading similarity statements as congruency statements
• incorrectly creating proportions between corresponding sides
Example 1

Prove \( \triangle ABC \sim \triangle DEC \).

1. Identify the given information.
   Lengths for each side of both triangles are given.
   
   \[
   \begin{align*}
   AB &= 4 & DE &= 6 \\
   BC &= 6 & EC &= 9 \\
   AC &= 8 & DC &= 12
   \end{align*}
   \]

2. Compare the side lengths of both triangles.
   Pair the lengths of the sides of \( \triangle ABC \) with the corresponding lengths of the sides of \( \triangle DEC \) to determine if there is a common ratio.
   
   \[
   \begin{align*}
   \frac{AB}{DE} &= \frac{4}{6} = \frac{2}{3} & \frac{BC}{EC} &= \frac{6}{9} = \frac{2}{3} & \frac{AC}{DC} &= \frac{8}{12} = \frac{2}{3}
   \end{align*}
   \]
   Notice the common ratio, \( \frac{2}{3} \); the side lengths are proportional.

3. State your conclusion.
   Similar triangles must have side lengths that are proportional.
   \( \triangle ABC \sim \triangle DEC \) by the Side-Side-Side (SSS) Similarity Statement.
Example 2

Determine whether the triangles are similar. Explain your reasoning.

1. Identify the given information.
   According to the diagram, \( \angle A \cong \angle E \).
   Given the side lengths, both \( \angle A \) and \( \angle E \) are included angles.

2. Compare the given side lengths of both triangles.
   If the triangles are similar, then the corresponding sides are proportional.
   
   \[
   \frac{AB}{EF} = \frac{6}{4} = \frac{3}{2} \quad \frac{AC}{ED} = \frac{10.5}{7} = \frac{3}{2}
   \]
   The side lengths are proportional.

3. State your conclusion.
   The measures of two sides of \( \triangle ABC \) are proportional to the measures of two corresponding sides of \( \triangle EFD \), and the included angles are congruent.
   \( \triangle ABC \sim \triangle EFD \) by the Side-Angle-Side (SAS) Similarity Statement.
Example 3

Determine whether the triangles are similar. Explain your reasoning.

1. Identify the given information.
   The measures of each side of both triangles are given.

2. Compare the side lengths of both triangles.
   Pair the lengths of the sides of $\triangle ABC$ with the corresponding lengths of the sides of $\triangle DEF$ to determine if there is a common ratio.

   \[
   \frac{AB}{DE} = \frac{4.5}{6} = \frac{3}{4} \quad \frac{BC}{EF} = \frac{4.5}{7.5} = \frac{3}{5} \quad \frac{AC}{DF} = \frac{6}{10} = \frac{3}{5}
   \]

   Notice there is not a common ratio; therefore, the side lengths are not proportional.

3. State your conclusion.
   Similar triangles must have side lengths that are proportional.
   $\triangle ABC$ is not similar to $\triangle DEF$. 

$\checkmark$
Example 4

Identify the similar triangles and then find the value of $x$.

1. Identify the given information.
   \[ \angle A \text{ and } \angle D \text{ are both right angles; therefore, } \angle A \cong \angle D. \]
   \[ \overline{AB} \text{ and } \overline{DC} \text{ are corresponding sides.} \]
   \[ \overline{AE} \text{ and } \overline{DF} \text{ are corresponding sides.} \]
   \[ \triangle ABE \sim \triangle DCF \text{ by the Side-Angle-Side (SAS) Similarity Statement.} \]

2. Determine the scale factor of the triangle sides.
   \[ \frac{AB}{DC} = \frac{3}{6} = \frac{1}{2} \]
   The scale factor is $\frac{1}{2}$.

3. Find the length of $x$.
   \[ \frac{AE}{DF} = \frac{x + 2.5}{x + 2.5} = \frac{4.5}{2} = \frac{1}{2} \]
   Solve the proportion $\frac{4.5}{x + 2.5} = \frac{1}{2}$ for $x$.
   \[ (4.5)(2) = (1)(x + 2.5) \]
   Find the cross products.
   \[ 9 = x + 2.5 \]
   Simplify.
   \[ x = 6.5 \]
   Solve for $x$.

The length of $x$ is 6.5 units.
Problem-Based Task 1.7.1: Down, Down, Down

Gutters are designed to channel rainwater away from roofs, preventing water damage and leaks. Downspouts lead down from the gutters to the ground. The amount of downspout needed depends on the height of the house. If the downspout for the gutters is too short, the water can cause flooding. Blair is using a mirror to determine the height of his house so he can know how long of a downspout he needs to buy. He has placed the mirror on the ground and is standing far enough away from it that he can see the gutters of his house in the mirror. Blair, who is 5 foot 10 inches tall, measured his distance from the mirror as 105 inches. The distance from the mirror to the foundation of the house is 450 inches. Downspouts are often sold in 5-foot sections. How many sections of downspout does Blair need to purchase?
Problem-Based Task 1.7.1: Down, Down, Down

Coaching

a. How tall is Blair in inches?

b. Are the triangles in the diagram similar? Explain.

c. What are the characteristics of similar triangles?

d. Is it possible to determine the height of the house from the foundation to the gutters using the given information? Explain your answer.

e. What is the height of the house from the foundation to the gutters?

f. How many 5-foot sections of downspout are needed for Blair’s house?
Problem-Based Task 1.7.1: Down, Down, Down

Coaching Sample Responses

a. How tall is Blair in inches?

There are 12 inches in a foot. Blair is 5 feet 10 inches tall.

To determine his height in inches, multiply the number of feet tall he is by 12, and then add the number of inches.

\[ \text{height in inches} = 5(12) + 10 = 60 + 10 = 70 \]

Blair is 70 inches tall.

b. Are the triangles in the diagram similar? Explain.

The triangles in the diagram are similar because of the Angle-Angle (AA) Similarity Statement. This statement asserts that if two angles of one triangle are congruent to two angles of another triangle, then the triangles are similar.

c. What are the characteristics of similar triangles?

The angles of similar triangles are congruent and the sides are proportional.
d. Is it possible to determine the height of the house from the foundation to the gutters using the given information? Explain your answer.

Yes, it is possible to determine the height of the house from the foundation to the gutters using the given information. You can do so by creating a proportion using the two similar triangles.

e. What is the height of the house from the foundation to the gutters?

Create a proportion to find the height of the house from the foundation to the gutters.

\[
\frac{\text{Blair’s height}}{\text{distance from mirror to Blair}} = \frac{\text{height from foundation to gutters}}{\text{distance from mirror to foundation}}
\]

\[
\frac{70}{x} = \frac{105}{450} \quad \text{Substitute.}
\]

\[(70)(450) = (105)(x) \quad \text{Find the cross products.}
\]

\[31,500 = 105x \quad \text{Simplify.}
\]

\[x = 300 \quad \text{Solve for } x.
\]

The height of the house from the foundation to the gutters is 300 inches.

f. How many 5-foot sections of downspout are needed for Blair’s house?

Determine the length of the needed downspout in feet.

Divide the number of inches by 12.

length of needed downspout in feet = \(\frac{300}{12} = 25\)

The length of the needed downspout in feet is 25 feet.

Each section of downspout is 5 feet.

Divide the number of feet needed by 5 to determine the number of sections needed.

number of sections needed = \(\frac{25}{5} = 5\)

Blair needs 5 sections of downspout.

**Recommended Closure Activity**

Select one or more of the essential questions for a class discussion or as a journal entry prompt.
Practice 1.7.1: Proving Triangle Similarity Using Side-Angle-Side (SAS) and Side-Side-Side (SSS) Similarity

Prove that the triangles are similar.

1.

2.

3.
Determine whether the triangles are similar. If the triangles are similar, write a similarity statement.

4. \( \triangle ADE \) and \( \triangle BCF \)

5. \( \triangle ABD \) and \( \triangle ECD \)

6. \( \triangle ABE \) and \( \triangle CDF \)

7. \( \triangle ACD \) and \( \triangle BCD \)
For problems 8–10, find $x$.

8.

9.

10.
Lesson 1.7.2: Working with Ratio Segments

Warm-Up 1.7.2

Landscapers will often stake a sapling to strengthen the tree’s root system. A typical method of staking a tree is to tie wires to both sides of the tree and then stake the wires to the ground. If done properly, the two stakes will be the same distance from the tree.

1. Assuming the distance from the tree trunk to each stake is equal, what is the value of $x$?

2. How far is each stake from the tree?

3. Assuming the distance from the tree trunk to each stake is equal, what is the value of $y$?

4. What is the length of the wire from each stake to the tie on the tree?
Lesson 1.7.2: Working with Ratio Segments

Common Core Georgia Performance Standard

MCC9–12.G.SRT.4

Warm-Up 1.7.2 Debrief

Landscapers will often stake a sapling to strengthen the tree’s root system. A typical method of staking a tree is to tie wires to both sides of the tree and then stake the wires to the ground. If done properly, the two stakes will be the same distance from the tree.

1. Assuming the distance from the tree trunk to each stake is equal, what is the value of $x$?

To find the value of $x$, set both expressions equal to each other.
4x – 5 = 2x + 1  
Set the expressions equal to each other.

2x – 5 = 1  
Subtract 2x from both sides of the equation.

2x = 6  
Add 5 to both sides of the equation.

x = 3  
Divide both sides by 3.

The value of x is 3.

2. How far is each stake from the tree?

To determine how far each stake is from the tree, substitute 3 for x in either expression.

4x – 5  
Original expression

4(3) – 5  
Substitute 3 for x.

12 – 5 = 7  
Simplify.

2x + 1  
Original expression

2(3) + 1  
Substitute 3 for x.

6 + 1 = 7  
Simplify.

Each stake is 7 units from the tree trunk.

3. Assuming the distance from the tree trunk to each stake is equal, what is the value of y?

To find the value of y, set both expressions equal to each other.

2y + 17 = 7y – 3  
Set the expressions equal to each other.

17 = 5y – 3  
Subtract 2y from both sides of the equation.

20 = 5y  
Add 3 to both sides of the equation.

y = 4  
Divide both sides by 5.

The value of y is 4.
4. What is the length of the wire from each stake to the tie on the tree?

To determine the length of the wire from each stake to the tie on the tree, substitute 4 for $y$ in each expression.

\[
\begin{align*}
2y + 17 & \quad \text{Original expression} \\
2(4) + 17 & \quad \text{Substitute 4 for } y. \\
8 + 17 = 25 & \quad \text{Simplify.}
\end{align*}
\]

\[
\begin{align*}
7y - 3 & \quad \text{Original expression} \\
7(4) - 3 & \quad \text{Substitute 4 for } y. \\
28 - 3 = 25 & \quad \text{Simplify.}
\end{align*}
\]

The length of the wire from each stake to the tie on the tree trunk is 25 units.

**Connection to the Lesson**

- Students will continue their work with triangles related to this example and expand on their understanding of similar triangles, specifically the relationship between angle bisectors and ratio segments.
Prerequisite Skills
This lesson requires the use of the following skills:

- creating ratios
- solving proportions
- identifying both corresponding and congruent parts of triangles
- understanding angle bisectors

Introduction
Archaeologists, among others, rely on the Angle-Angle (AA), Side-Angle-Side (SAS), and Side-Side-Side (SSS) similarity statements to determine actual distances and locations created by similar triangles. Many engineers, surveyors, and designers use these statements along with other properties of similar triangles in their daily work. Having the ability to determine if two triangles are similar allows us to solve many problems where it is necessary to find segment lengths of triangles.

Key Concepts

- If a line parallel to one side of a triangle intersects the other two sides of the triangle, then the parallel line divides these two sides proportionally.
- This is known as the Triangle Proportionality Theorem.

Theorem

Triangle Proportionality Theorem

If a line parallel to one side of a triangle intersects the other two sides of the triangle, then the parallel line divides these two sides proportionally.

- In the figure above, \( \overline{AC} \parallel \overline{DE} \); therefore, \( \frac{AD}{DB} = \frac{CE}{EB} \).
• Notice the arrows in the middle of $\overline{DE}$ and $\overline{AC}$, which indicate the segments are parallel.

• This theorem can be used to find the lengths of various sides or portions of sides of a triangle.

• $\triangle ABC \sim \triangle DBE$ because of the Side-Angle-Side (SAS) Similarity Statement.

• It is also true that if a line divides two sides of a triangle proportionally, then the line is parallel to the third side.

In the figure above, $\frac{AD}{DB} = \frac{CE}{EB}$; therefore, $\overline{AC} \parallel \overline{DE}$.

• This is helpful when determining if two lines or segments are parallel.

• It is possible to determine the lengths of the sides of triangles because of the **Segment Addition Postulate**.

• This postulate states that if $B$ is between $A$ and $C$, then $AB + BC = AC$.

• It is also true that if $AB + BC = AC$, then $B$ is between $A$ and $C$.

• Segment congruence is also helpful when determining the lengths of sides of triangles.

• The **Reflexive Property of Congruent Segments** means that a segment is congruent to itself, so $\overline{AB} \cong \overline{AB}$.

• According to the **Symmetric Property of Congruent Segments**, if $\overline{AB} \cong \overline{CD}$, then $\overline{CD} \cong \overline{AB}$.

• The **Transitive Property of Congruent Segments** allows that if $\overline{AB} \cong \overline{CD}$ and $\overline{CD} \cong \overline{EF}$, then $\overline{AB} \cong \overline{EF}$.
This information is also helpful when determining segment lengths and proving statements.

If one angle of a triangle is bisected, or cut in half, then the **angle bisector** of the triangle divides the opposite side of the triangle into two segments that are proportional to the other two sides of the triangle.

This is known as the Triangle Angle Bisector Theorem.

### Theorem

**Triangle Angle Bisector Theorem**

If one angle of a triangle is bisected, or cut in half, then the angle bisector of the triangle divides the opposite side of the triangle into two segments that are proportional to the other two sides of the triangle.

In the figure above, \( \angle ABD \cong \angle DBC \); therefore, \( \frac{AD}{DC} = \frac{BA}{BC} \).

These theorems can be used to determine segment lengths as well as verify that lines or segments are parallel.

### Common Errors/Misconceptions

- assuming a line parallel to one side of a triangle bisects the remaining sides rather than creating proportional sides
- interchanging similarity statements with congruence statements
Guided Practice 1.7.2

Example 1

Find the length of $BE$.

1. Identify the given information.

   According to the diagram, $AC \parallel DE$.

2. Find the length of $BE$.

   Use the Triangle Proportionality Theorem to find the length of $BE$.

   $$\frac{BD}{DA} = \frac{BE}{EC}$$

   $5.5 \quad x$

   $2 \quad 3$

   $(3)(5.5) = (2)(x)$

   $16.5 = 2x$

   $x = 8.25$

   The length of $BE$ is 8.25 units.
Example 2

Using two different methods, find the length of $CA$.

Method 1

1. Identify the given information.

   According to the diagram, $AB \parallel DE$.

2. Find the length of $CA$.

   Use the Triangle Proportionality Theorem to find the length of $CA$.

   $\frac{CA}{CD} = \frac{CB}{CE}$

   $\frac{CA}{5} = \frac{9 + 6}{6}$

   Create a proportion.

   Substitute the known lengths of each segment.

   $(6)(CA) = (5)(9 + 6)$

   Find the cross products.

   $(6)(CA) = (5)(15)$

   Solve for $CA$.

   $(6)(CA) = 75$

   $CA = 12.5$

   The length of $CA$ is 12.5 units.
Method 2

1. Identify the given information.

According to the diagram, \( \overline{AB} \parallel \overline{DE} \).

2. Find the length of \( \overline{CA} \).

Use the Triangle Proportionality Theorem to find the length of \( \overline{CA} \).

An alternative method for finding the length of \( \overline{CA} \) is to first find the length of \( \overline{DA} \).

\[
\frac{CD}{DA} = \frac{CE}{EA}
\]

Create a proportion.

\[
\frac{5}{x} = \frac{6}{9}
\]

Substitute the known lengths of each segment.

\[
(5)(9) = (6)(x)
\]

Find the cross products.

\[
45 = 6x
\]

Solve for \( x \).

\[
x = 7.5
\]

The length of \( \overline{CA} \) is equal to the sum of \( \overline{CD} \) and \( \overline{DA} \).

\[
\overline{CD} + \overline{DA} = \overline{CA}
\]

\[
5 + 7.5 = 12.5
\]

The length of \( \overline{CA} \) is 12.5 units.
Example 3

Prove that $DE \parallel AC$.

1. Determine if $DE$ divides $BA$ and $BC$ proportionally.
   
   \[
   \frac{BD}{DA} = \frac{9}{3} = 3 \quad \quad \frac{BE}{EC} = \frac{10.5}{3.5} = 3
   \]

2. State your conclusion.
   
   \[
   \frac{BD}{DA} = \frac{BE}{EC} = 3; \text{ therefore, } DE \parallel AC \text{ because of the Triangle Proportionality Theorem.}
   \]
Example 4
Is $DE \parallel AC$?

1. Determine if $DE$ divides $BA$ and $BC$ proportionally.

$\frac{BD}{DA} = \frac{11}{7}$

$\frac{BE}{EC} = \frac{5.25}{3.5} = \frac{3}{2}$

2. State your conclusion.

$\frac{11}{7} \neq \frac{3}{2}$

$\frac{BD}{DA} \neq \frac{BE}{EC}$; therefore, $DE$ is not parallel to $AC$ because of the Triangle Proportionality Theorem.
Example 5

Find the lengths of $BD$ and $DC$.

1. Identify the given information.
   
   $\angle BAD \cong \angle DAC$
   
   $\frac{BD}{BA} = \frac{BA}{AC}$ because of the Triangle Angle Bisector Theorem.

2. Determine the lengths of $BD$ and $DC$.
   
   Create a proportion.
   
   $\frac{BD}{BA} = \frac{BA}{AC}$
   
   $\frac{x+2}{x+1} = \frac{7}{5.6}$
   
   Substitute the known lengths of each segment.
   
   $(x+2)(5.6) = (x+1)(7)$
   
   Find the cross products.
   
   $5.6x + 11.2 = 7x + 7$
   
   Solve for $x$.
   
   $4.2 = 1.4x$
   
   $x = 3$

   $BD = x + 2 = 3 + 2 = 5$
   
   $DC = x + 1 = 3 + 1 = 4$

3. State your conclusion.
   
   The length of $BD$ is 5 units.
   
   The length of $DC$ is 4 units.
Problem-Based Task 1.7.2: Suddenly Sinking

In August 2012, a sinkhole in Louisiana swallowed part of the Earth’s surface. This sudden erosion of land forced many people to evacuate as engineers worked to determine the effects of the sinkhole. Officials began their assessment of the area by first determining the size of the sinkhole. Surveyors worked to locate points near the sinkhole to calculate its diameter. The initial measurements are shown below. As days passed, the diameter of the sinkhole increased by 15%. What was the diameter of the sinkhole after the increase?
Problem-Based Task 1.7.2: Suddenly Sinking

Coaching

a. Are the triangles in the diagram similar? Explain.

b. What are the characteristics of similar triangles?

c. Is it possible to determine the diameter of the initial sinkhole using the given information? Explain your answer.

d. Determine the length of $DE$.

e. What was the diameter of the initial sinkhole?

f. How are percent increases calculated?

g. What is 15% of the initial diameter?

h. What was the diameter after the sinkhole increased?
Problem-Based Task 1.7.2: Suddenly Sinking

Coaching Sample Responses

a. Are the triangles in the diagram similar? Explain.

The triangles in the diagram are similar. \( BC \) is parallel to \( DE \) of \( \triangle ADE \).

According to the Triangle Proportionality Theorem, if a line parallel to one side of a triangle intersects the other two sides of the triangle, then the parallel line divides these two sides proportionally.

b. What are the characteristics of similar triangles?

The angles of similar triangles are congruent and the sides are proportional.

c. Is it possible to determine the diameter of the initial sinkhole using the given information? Explain your answer.

Yes, it is possible to determine the diameter of the initial sinkhole using the given information. You can do so by creating a proportion using the smaller similar triangle, \( \triangle ABC \).

d. Determine the length of \( DE \).

Create a proportion to find the length of \( DE \).

\[
\frac{AD}{DE} = \frac{AB}{BC}
\]

Create a proportion.

\[
\frac{336 + 192}{x} = \frac{336}{194}
\]

Substitute known values.

\[
\frac{528}{x} = \frac{336}{194}
\]

Simplify.

\[
(528)(194) = (336)(x)
\]

Find the cross products.

\[
102,432 = 336x
\]

Simplify.

\[
x \approx 305 \text{ feet}
\]

The length of \( DE \) is approximately 305 feet.

e. What was the diameter of the initial sinkhole?

The diameter of the initial sinkhole was approximately 305 feet.
f. How are percent increases calculated?

One way to calculate a percent increase is to add the initial value to the product of the initial value and the decimal equivalent of the percent.

\[
\text{initial value} + \frac{x}{100} (\text{initial value})
\]

g. What is 15\% of the initial diameter?

15\% of 305 can be written numerically as 0.15(305).

The result is 45.75.

h. What was the diameter after the sinkhole increased?

Add 15\% of the initial diameter to the initial diameter.

\[
45.75 + 305 = 350.75
\]

The diameter of the sinkhole after the increase was 350.75 feet.

**Recommended Closure Activity**

Select one or more of the essential questions for a class discussion or as a journal entry prompt.
Practice 1.7.2: Working with Ratio Segments

Use the Triangle Proportionality Theorem and the Triangle Angle Bisector Theorem to find the unknown lengths of the given segments.

1. $\overline{BD}$

![Diagram of triangle with segments labeled BD, CD, and AD.]

2. $\overline{BE}$

![Diagram of triangle with segments labeled BE, DE, and AD.]

3. $\overline{EC}$

![Diagram of triangle with segments labeled EC, ED, and AD.]

continued
4. $\overline{BD}$

5. $\overline{CD}; \overline{BD}$

6. $\overline{CB}; \overline{CD}$
Unit 1 • Similarity, Congruence, and Proofs

Lesson 7: Proving Similarity

Use the Triangle Proportionality Theorem to determine if the given segments are parallel. Explain your reasoning.

7. Is $\overline{BE} \parallel \overline{CD}$?

8. Is $\overline{AB} \parallel \overline{EC}$?
9. If \( AC = 60 \) units and \( EC = 36 \) units, is \( \overline{AE} \parallel \overline{BD} \)?

10. If \( AC = 24 \) units and \( AD = 30 \) units, is \( \overline{BE} \parallel \overline{CD} \)?
Lesson 1.7.3: Proving the Pythagorean Theorem Using Similarity

Warm-Up 1.7.3

Woodworkers must accurately cut and assemble each piece of wood to ensure that a project is “square.” Every vertical piece should intersect every horizontal piece at a 90° angle. To determine if a project is square, woodworkers use the Pythagorean Theorem, which states that the sum of the squares of the two legs of a right triangle is equal to the square of the longest side. If the lengths of the diagonals are equal, then the project is square. Use the diagram below of a door to solve the problems that follow.

1. A woodworker measured the length of one diagonal of the wooden door, $BD$, to be 212 cm. The woodworker measured the length of $AD$ to be 198 cm and the length of $DC$ to be 76 cm. Calculate the length of $AC$.
2. Is $BD$ congruent to $AC$?
3. Is the door “square”? Explain your answer.
Lesson 1.7.3: Proving the Pythagorean Theorem Using Similarity

Common Core Georgia Performance Standard
MCC9–12.G.SRT.4

Warm-Up 1.7.3 Debrief

Woodworkers must accurately cut and assemble each piece of wood to ensure that a project is “square.” Every vertical piece should intersect every horizontal piece at a 90° angle. To determine if a project is square, woodworkers use the Pythagorean Theorem, which states that the sum of the squares of the two legs of a right triangle is equal to the square of the longest side. If the lengths of the diagonals are equal, then the project is square. Use the diagram below of a door to solve the problems that follow.

1. A woodworker measured the length of one diagonal of the wooden door, $BD$, to be 212 cm. The woodworker measured the length of $AD$ to be 198 cm and the length of $DC$ to be 76 cm. Calculate the length of $AC$. 
**UNIT 1 • SIMILARITY, CONGRUENCE, AND PROOFS**

Lesson 7: Proving Similarity

**Instruction**

\( \overline{AD} \) and \( \overline{DC} \) are the two shorter sides of \( \triangle ADC \).

Use the Pythagorean Theorem to calculate the length of \( \overline{AC} \).

\[ a^2 + b^2 = c^2 \]  
Pythagorean Theorem

\[ (198)^2 + (76)^2 = c^2 \]  
Substitute values for \( a \) and \( b \).

\[ 39,204 + 5776 = c^2 \]  
Simplify.

\[ 44,980 = c^2 \]  
Take the positive square root of both sides.

\[ c = \sqrt{44,980} \approx 212.1 \text{ cm} \]

The length of \( \overline{AC} \) is \( \sqrt{44,980} \) cm, or approximately 212.1 cm.

2. Is \( \overline{BD} \) congruent to \( \overline{AC} \)?

The length of \( \overline{BD} \) was measured to be 212 cm.

The calculated length of \( \overline{AC} \) is \( \sqrt{44,980} \) cm, or approximately 212.1 cm.

Although the lengths are close, they are not congruent.

3. Is the door “square”? Explain your answer.

To be square, the lengths of each diagonal must be congruent.

The lengths are not congruent; therefore, the door, as assembled, is not square.

**Connection to the Lesson**

- Students have worked with the Pythagorean Theorem in the past and will deepen their understanding of this theorem while determining similar triangles.
Prerequisite Skills

This lesson requires the use of the following skills:

- using the distance formula to find the lengths of sides of triangles
- being familiar with the Pythagorean Theorem
- working with and simplifying square roots
- identifying similar triangles
- using similarity statements to find unknown lengths and measures of similar triangles

Introduction

Geometry includes many definitions and statements. Once a statement has been shown to be true, it is called a theorem. Theorems, like definitions, can be used to show other statements are true. One of the most well known theorems of geometry is the Pythagorean Theorem, which relates the length of the hypotenuse of a right triangle to the lengths of its legs. The theorem states that the sum of the squares of the lengths of the legs \((a\) and \(b)\) of a right triangle is equal to the square of the length of the hypotenuse \((c)\). This can be written algebraically as \(a^2 + b^2 = c^2\). The Pythagorean Theorem has many applications and can be very helpful when solving real-world problems. There are several ways to prove the Pythagorean Theorem; one way is by using similar triangles and similarity statements.

Key Concepts

The Pythagorean Theorem

- The Pythagorean Theorem is often used to find the lengths of the sides of a right triangle, a triangle that includes one 90° angle.

<table>
<thead>
<tr>
<th>Theorem</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Pythagorean Theorem</strong></td>
</tr>
<tr>
<td>The sum of the squares of the lengths of the legs ((a) and (b)) of a right triangle is equal to the square of the length of the hypotenuse ((c)).</td>
</tr>
</tbody>
</table>

\[ a^2 + b^2 = c^2 \]
• In the triangle on the previous page, angle \( C \) is \( 90^\circ \), as shown by the square.

• The longest side of the right triangle, \( c \), is called the hypotenuse and is always located across from the right angle.

• The legs of the right triangle, \( a \) and \( b \), are the two shorter sides.

• It is also true that if the sum of the squares of the measures of two sides of a triangle equals the square of the measure of the longest side, then the triangle is a right triangle.

• This is known as the converse of the Pythagorean Theorem.

• To prove the Pythagorean Theorem using similar triangles, you must first identify the similar triangles.

• In this example, there is only one triangle given.

• Begin by drawing the altitude, the segment from angle \( C \) that is perpendicular to the line containing the opposite side, \( c \).

Notice that by creating the altitude \( \overline{CD} \), we have created two smaller right triangles, \( \triangle ADC \) and \( \triangle BDC \), within the larger given right triangle, \( \triangle ACB \).

• \( \angle ACB \) and \( \angle ADC \) are \( 90^\circ \) and are therefore congruent.

• \( \angle A \) of \( \triangle ADC \) is congruent to \( \angle A \) of \( \triangle ACB \) because of the Reflexive Property of Congruence.
• According to the Angle-Angle (AA) Similarity Statement, if two angles of one triangle are congruent to two angles of another triangle, then the triangles are similar; therefore, \( \triangle ADC \sim \triangle ACB \).

• \( \angle ACB \) and \( \angle BDC \) are 90° and are therefore congruent.

• \( \angle B \) of \( \triangle BDC \) is congruent to \( \angle B \) of \( \triangle ACB \) because of the Reflexive Property of Congruence.

• Two angles in \( \triangle BDC \) are congruent to two angles in \( \triangle ACB \); therefore, \( \triangle BDC \sim \triangle ACB \).

• Similarity is transitive. Since \( \triangle ADC \sim \triangle ACB \) and \( \triangle BDC \sim \triangle ACB \), then \( \triangle ADC \sim \triangle BDC \).

• Corresponding sides of similar triangles are proportional; therefore, \( \frac{c}{b} = \frac{a}{d} \) and \( \frac{c}{a} = \frac{e}{d} \).

• Determining the cross products of each proportion leads to the Pythagorean Theorem.

\[
\begin{align*}
    \frac{c}{b} &= \frac{a}{d} \\
    \frac{c}{a} &= \frac{e}{d} \\
    cd &= b^2 \\
    ce &= a^2
\end{align*}
\]

\[
\begin{align*}
    cd + ce &= a^2 + b^2 & \text{Add both equations.} \\
    c(e + d) &= a^2 + b^2 & \text{Factor.} \\
    c^2 &= a^2 + b^2 & (e + d) \text{ is equal to } c \text{ because of segment addition.}
\end{align*}
\]
• The converse of the Pythagorean Theorem can be useful when proving right triangles using similar triangles.

Types of Proofs
• **Paragraph proofs** are statements written out in complete sentences in a logical order to show an argument.
• **Flow proofs** are a graphical method of presenting the logical steps used to show an argument.
  • In a flow proof, the logical statements are written in boxes and the reason for each statement is written below the box.
• Another accepted form of proof is a **two-column proof**.
  • Two-column proofs include numbered statements and corresponding reasons that show the argument in a logical order.
  • Two-column proofs appear in the Guided Practice examples that follow.

Common Errors/Misconceptions
• misidentifying the altitudes of triangles
• incorrectly simplifying expressions with square roots
Example 1

Write a two-column proof to prove the Pythagorean Theorem using similar triangles.

1. Draw in helpful information.
   Draw the altitude from \( \angle C \) and label the point of intersection on \( \overline{AB} \) as \( D \).
   Label \( AD \) as \( e \).
   Label \( DB \) as \( f \).
2. Identify the similar triangles.

It is often helpful to redraw the triangles.

3. Create the two-column proof.

<table>
<thead>
<tr>
<th>Statements</th>
<th>Reasons</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. ( \triangle ABC ) with right ( \angle C )</td>
<td>1. Given</td>
</tr>
<tr>
<td>2. ( \triangle ABC \sim \triangle ACD ) ( \triangle ABC \sim \triangle CBD )</td>
<td>2. If the altitude is drawn to the hypotenuse of a right triangle, then the two triangles formed are similar to the original triangle and each other.</td>
</tr>
<tr>
<td>3. ( \frac{c}{a} = \frac{a}{f}; \frac{c}{b} = \frac{b}{e} )</td>
<td>3. Definition of similar triangles; corresponding sides are proportional.</td>
</tr>
<tr>
<td>4. ( cf = a^2; ce = b^2 )</td>
<td>4. Multiplication Property of Equality</td>
</tr>
<tr>
<td>5. ( cf + ce = a^2 + b^2 )</td>
<td>5. Addition Property of Equality</td>
</tr>
<tr>
<td>6. ( c(f + e) = a^2 + b^2 )</td>
<td>6. Distributive Property of Equality</td>
</tr>
<tr>
<td>7. ( e + f = c )</td>
<td>7. Segment Addition Postulate</td>
</tr>
<tr>
<td>8. ( c(c) = a^2 + b^2 ) or ( c^2 = a^2 + b^2 )</td>
<td>8. Substitution Property</td>
</tr>
</tbody>
</table>
Example 2

Find the length of the altitude, $x$, of $\triangle ABC$.

1. Identify the similar triangles.

   $\triangle ABC$ is a right triangle.

   The altitude of $\triangle ABC$ is drawn from right $\angle ACB$ to the opposite side, creating two smaller similar triangles.

   $\triangle ABC \sim \triangle ACD \sim \triangle CBD$

2. Use corresponding sides to write a proportion containing $x$.

   \[
   \frac{\text{shorter leg of } \triangle ACD}{\text{shorter leg of } \triangle CBD} = \frac{\text{longer leg of } \triangle ACD}{\text{longer leg of } \triangle CBD}
   \]

   \[
   \frac{x}{10} = \frac{18}{x}
   \]

   Substitute values for each side.

   $(x)(x) = (10)(18)$ Find the cross products.

   $x^2 = 180$ Simplify.

   $x = 6\sqrt{5} \approx 13.4$ Take the positive square root of each side.

3. Summarize your findings.

   The length of the altitude, $x$, of $\triangle ABC$ is $6\sqrt{5}$ units, or approximately 13.4 units.
Example 3

Find the unknown values in the figure.

1. Identify the similar triangles.
   \( \triangle ABC \) is a right triangle.
   The altitude of \( \triangle ABC \) is drawn from right \( \angle ACB \) to the opposite side, creating two smaller similar triangles.
   \( \triangle ABC \sim \triangle ACD \sim \triangle CBD \)

2. Use corresponding sides to write a proportion containing \( c \).
   \[
   \frac{\text{hypotenuse of } \triangle ACD}{\text{hypotenuse of } \triangle ABC} = \frac{\text{shorter leg of } \triangle ACD}{\text{shorter leg of } \triangle ABC}
   \]
   \[
   \frac{8}{c} = \frac{4.8}{6}
   \]
   Substitute values for each side.
   \[
   (8)(6) = (c)(4.8)
   \]
   Find the cross products.
   \[
   48 = 4.8c
   \]
   Simplify.
   \[
   c = 10
   \]
   Solve for \( c \).
UNIT 1 • SIMILARITY, CONGRUENCE, AND PROOFS
Lesson 7: Proving Similarity

3. Use corresponding sides to write a proportion containing $e$.
   
   \[
   \frac{\text{longer leg of } \triangle ACD}{\text{longer leg of } \triangle ABC} = \frac{\text{shorter leg of } \triangle ACD}{\text{shorter leg of } \triangle ABC}
   \]

   \[
   \frac{e}{8} = \frac{4.8}{6}
   \]
   Substitute values for each side.

   \[
   (e)(6) = (8)(4.8)
   \]
   Find the cross products.

   \[
   6e = 38.4
   \]
   Simplify.

   \[
   e = 6.4
   \]
   Solve for $e$.

4. Use corresponding sides to write a proportion containing $f$.
   
   \[
   \frac{\text{longer leg of } \triangle CBD}{\text{longer leg of } \triangle ABC} = \frac{\text{shorter leg of } \triangle CBD}{\text{shorter leg of } \triangle ABC}
   \]

   \[
   \frac{4.8}{8} = \frac{f}{6}
   \]
   Substitute values for each side.

   \[
   (4.8)(6) = (8)(f)
   \]
   Find the cross products.

   \[
   28.8 = 8f
   \]
   Simplify.

   \[
   f = 3.6
   \]
   Solve for $f$.

5. Summarize your findings.
   
   The length of $c$ is 10 units.
   The length of $e$ is 6.4 units.
   The length of $f$ is 3.6 units.
Problem-Based Task 1.7.3: Towering Heights

Cell towers are tall structures that provide service to customers with cell phones, smartphones, tablets, and other wireless communication devices. As the use of wireless devices increases, the need for cell towers goes up. According to the Federal Aviation Administration (FAA) and the Federal Communications Commission (FCC), any structure taller than 61 meters must be lit up so aircraft pilots can see it.

A cell phone company is interested in placing a 5-meter-tall antenna on an existing tower in order to boost their cell signal without having to build a new tower. A surveyor standing 7.5 meters from the base of the tower calculates the height of the existing tower. The line of sight from the lens on the surveyor’s tripod to the base of the tower is 1 meter above the ground. If the 5-meter-tall antenna is added to the top of the tower, will the company be required to add lighting to the existing structure? Explain your reasoning.
Problem-Based Task 1.7.3: Towering Heights

Coaching

a. Are the triangles in the diagram similar? Explain.

b. Write a statement indicating similar triangles.

c. What are the characteristics of similar triangles?

d. Is it possible to determine the height of the tower using the given information? Explain your answer.

e. Determine the length of $\overline{AD}$.

f. What is the height of the tower?

g. How tall will the tower be once the antenna is added?

h. Will the height of the tower with the antenna exceed the maximum height stated by the FAA and FCC?

i. Will the company be required to add lighting to the tower once the antenna is added?
Problem-Based Task 1.7.3: Towering Heights

Coaching Sample Responses

a. Are the triangles in the diagram similar? Explain.
   Yes, the triangles are similar. The triangle created from the base of the tower to the height of the tower to the surveyor is a right triangle. The segment from the 90° angle to the tower is perpendicular to the tower and is the altitude of the triangle. By definition, the altitude of the right triangle creates two smaller, similar right triangles.

b. Write a statement indicating similar triangles.
   \( \triangle ABC \sim \triangle CDB \sim \triangle ADC \)

c. What are the characteristics of similar triangles?
   The angles of similar triangles are congruent and the sides are proportional.

d. Is it possible to determine the height of the tower using the given information? Explain your answer.
   Yes, it is possible to determine the height of the tower using the given information by creating a proportion using the two smaller similar triangles, \( \triangle CDB \) and \( \triangle ADC \).

e. Determine the length of \( AD \).
   Create a proportion to find the length of \( AD \).
   \[
   \begin{align*}
   \frac{CD}{BD} &= \frac{AD}{CD} \\
   \frac{7.5}{1} &= \frac{x}{7.5}
   \end{align*}
   \]
   Substitute known values.
   \[
   (7.5)(7.5) = (1)(x)
   \]
   Find the cross products.
   \[
   56.25 = x
   \]
   Simplify.
   The length of \( AD \) is 56.25 meters.
f. What is the height of the tower?
   To find the height of the tower, add the length of $\overline{AD}$ to the length of $\overline{BD}$.
   
   $$\overline{AD} + \overline{BD} = 56.25 + 1 = 57.25$$
   
   The height of the tower is 57.25 meters.

   
g. How tall will the tower be once the antenna is added?
   Add the height of the antenna to the height of the tower.
   
   $$5 + 57.25 = 62.25$$
   
   The tower will be 62.25 meters tall after the antenna is added.

   
h. Will the height of the tower with the antenna exceed the maximum height stated by the FAA and FCC?
   Yes; 62.25 is greater than 61 meters, the maximum height stated by the FAA and FCC.

   
i. Will the company be required to add lighting to the tower once the antenna is added?
   Yes; the tower with the antenna will exceed 61 meters, so lighting will have to be added.

   
**Recommended Closure Activity**

Select one or more of the essential questions for a class discussion or as a journal entry prompt.
Practice 1.7.3: Proving the Pythagorean Theorem Using Similarity

Find the unknown length(s) in each figure.

1.

![Diagram of a triangle with labels A, D, B, C, and x, where AD = 4, DC = 9, and AC = x.]

2.

![Diagram of another triangle with labels B, D, C, A, and x, where BD = 5, DC = x, and CA = 20.]

continued
3. \[ \triangle ABC \]

\[ \frac{AB}{CA} = \frac{BD}{CD} \]

4. \[ \triangle ACD \]

\[ \frac{AD}{CD} = \frac{x}{31.2} \]

\[ AD = 5 \]

continued
Lesson 7: Proving Similarity

5.

6.
7. Given triangles $\triangle ABC$ and $\triangle DEF$ with sides $AC = 8$ and $AD = 16$.

8. Triangle $\triangle ABC$ with sides $AB = 16$ and $AD = 6$.
10. Using similar triangles, write a two-column proof to prove the converse of the Pythagorean Theorem.

Given: \( \triangle ABC \), with \( c^2 = a^2 + b^2 \)

Prove: \( \triangle ABC \) is a right triangle.
Lesson 1.7.4: Solving Problems Using Similarity and Congruence

Warm-Up 1.7.4

Three buildings border a triangular courtyard as shown in the diagram. A walkway runs parallel to the edge of the courtyard labeled $\overline{CE}$. Landscapers would like to install a picket fence along the outside of the courtyard with the exception of the walkway. The fencing comes in 8-foot lengths.

1. Identify the similar triangles.

2. While preparing the sketch of the courtyard, landscapers forgot to measure the length of the courtyard represented by $\overline{DE}$. What is the length of $\overline{DE}$?

3. How many sections of fencing are needed?
Solving Problems Using Similarity and Congruence

Common Core Georgia Performance Standard
MCC9–12.G.SRT.5

Warm-Up 1.7.4 Debrief

Three buildings border a triangular courtyard as shown in the diagram. A walkway runs parallel to the edge of the courtyard labeled $\overline{CE}$. Landscapers would like to install a picket fence along the outside of the courtyard with the exception of the walkway. The fencing comes in 8-foot lengths.

1. Identify the similar triangles.

The walkway, $\overline{BD}$, is parallel to the edge of the courtyard labeled $\overline{CE}$. If a line is parallel to one side of a triangle and intersects the other two sides in two different points, then it separates these sides into segments of proportional lengths. The parallel line creates two similar triangles.

$\triangle ACE \sim \triangle ABD$
2. While preparing the sketch of the courtyard, landscapers forgot to measure the length of the courtyard represented by $DE$. What is the length of $DE$?

Use the Triangle Proportionality Theorem to find the length of $DE$.

$$\frac{AB}{BC} = \frac{AD}{DE}$$

Create a proportion.

$$\frac{410}{164} = \frac{595}{DE}$$

Substitute the known lengths of each segment.

$$(410)(DE) = (164)(595)$$

Find the cross products.

$$410(DE) = 97,580$$

Solve for $DE$.

$$DE = \frac{97,580}{410} = 238$$

The length of $DE$ is 238 feet.

3. How many sections of fencing are needed?

Calculate the perimeter of the courtyard excluding the walkway.

$$AB + BC + AD + DE$$

Equation for calculating perimeter

$$410 + 164 + 595 + 238$$

Substitute the lengths of each segment.

$$1407$$

Simplify.

The perimeter of the courtyard is 1,407 feet.

Determine the number of sections needed.

The fencing comes in 8-foot lengths.

Divide the total perimeter by 8.

$$1407 \div 8 = 175.875$$

Partial sections cannot be purchased, so 176 sections of fencing are needed.

**Connection to the Lesson**

- Students will continue to work with similar triangles. Students will apply the properties of similar triangles to solve for unknown values related to real-world contexts.
Introduction
Design, architecture, carpentry, surveillance, and many other fields rely on an understanding of the properties of similar triangles. Being able to determine if triangles are similar and understanding their properties can help you solve real-world problems.

Key Concepts
Similarity
• Similarity statements include Angle-Angle (AA), Side-Angle-Side (SAS), and Side-Side-Side (SSS).
• These statements allow us to prove triangles are similar.
• Similar triangles have corresponding sides that are proportional.
• It is important to note that while both similarity and congruence statements include an SSS and an SAS statement, the statements do not mean the same thing.
• Similar triangles have corresponding sides that are proportional, whereas congruent triangles have corresponding sides that are of the same length.

Triangle Theorems
• The Triangle Proportionality Theorem states that if a line parallel to one side of a triangle intersects the other two sides of the triangle, then the parallel line divides these two sides proportionally.
• This theorem can be used to find the lengths of various sides or portions of sides of a triangle.
• It is also true that if a line divides two sides of a triangle proportionally, then the line is parallel to the third side.
• The Triangle Angle Bisector Theorem states if one angle of a triangle is bisected, or cut in half, then the angle bisector of the triangle divides the opposite side of the triangle into two segments that are proportional to the other two sides of the triangle.

• The Pythagorean Theorem, written symbolically as $a^2 + b^2 = c^2$, is often used to find the lengths of the sides of a right triangle, which is a triangle that includes one $90^\circ$ angle.

• Drawing the altitude, the segment from the right angle perpendicular to the line containing the opposite side, creates two smaller right triangles that are similar.

**Common Errors/Misconceptions**

• misidentifying congruent angles because of the orientation of the triangles
• incorrectly creating proportions between corresponding sides
• assuming a line parallel to one side of a triangle bisects the remaining sides rather than creating proportional sides
• misidentifying the altitudes of triangles
• incorrectly simplifying expressions with square roots
Guided Practice 1.7.4

Example 1

A meterstick casts a shadow 65 cm long. At the same time, a tree casts a shadow 2.6 m long. How tall is the tree?

1. Draw a picture to understand the information.
2. Determine if the triangles are similar.

The rays of the sun create the shadows, which are considered to be parallel.

Right angles are formed between the ground and the meterstick as well as the ground and the tree.

Two angles of the triangles are congruent; therefore, by Angle-Angle Similarity, the triangles are similar.
3. Solve the problem.

Convert all measurements to the same units.

Similar triangles have proportional sides.

Create a proportion to find the height of the tree.

\[
\frac{\text{height of stick}}{\text{height of tree}} = \frac{\text{length of stick’s shadow}}{\text{length of tree’s shadow}}
\]

Create a proportion.

Substitute known values.

Find the cross products.

Simplify.

Solve for \( x \).

The height of the tree is 4 meters.
Example 2

Finding the distance across a canyon can often be difficult. A drawing of similar triangles can be used to make this task easier. Use the diagram to determine $AR$, the distance across the canyon.

1. Interpret the given information.
   A person standing at point $A$ can sight a rock across the canyon at point $R$.
   Point $C$ is selected so that $CA$ is perpendicular to $AR$, the distance across the canyon.
   Point $D$ is selected so that $CD$ is perpendicular to $CA$ and can be easily measured.
   The point of intersection of $RD$ and $CA$, point $B$, can then be found.
2. Determine if the triangles are similar.

\[ \angle A \text{ and } \angle C \text{ both measure } 90^\circ \text{ and are congruent.} \]

\[ \angle RBA \cong \angle DBC \]

By the Angle-Angle Similarity Statement, \( \triangle RBA \sim \triangle DBC \).

3. Solve the problem.

Similar triangles have proportional sides.

Create a proportion to find the distance across the canyon.

\[
\frac{AB}{BC} = \frac{AR}{CD}
\]

Create a proportion.

\[
\frac{180}{90} = \frac{x}{75}
\]

Substitute known values.

\[(180)(75) = (90)(x) \]

Find the cross products.

\[13,500 = 90x\]

Simplify.

\[x = 150\]

Solve for \(x\).

The distance across the canyon is 150 meters.
Example 3

To find the distance across a pond, Rita climbs a 30-foot observation tower on the shore of the pond and locates points $A$ and $B$ so that $\overline{AC}$ is perpendicular to $\overline{CB}$. She then finds the measure of $\overline{DB}$ to be 12 feet. What is the measure of $\overline{AD}$, the distance across the pond?

1. Determine if the triangles are similar.

   $\triangle ABC$ is a right triangle with $\angle C$ the right angle.
   $\overline{CD}$ is the altitude of $\triangle ABC$, creating two similar triangles, $\triangle ACD$ and $\triangle CBD$.

   $\triangle ABC \sim \triangle ACD \sim \triangle CBD$

2. Solve the problem.

   Similar triangles have proportional sides.

   Create a proportion to find the distance across the pond.

   \[
   \frac{BD}{CD} = \frac{CD}{AD}
   \]

   Create a proportion.

   \[
   \frac{12}{30} = \frac{30}{x}
   \]

   Substitute known values.

   \[
   (12)(x) = (30)(30)
   \]

   Find the cross products.

   \[
   12x = 900
   \]

   Simplify.

   \[
   x = 75
   \]

   Solve for $x$.

   The distance across the pond is 75 feet.
Example 4

To estimate the height of an overhang, a surveyor positions herself so that her line of sight to the top of the overhang and her line of sight to the bottom form a right angle. What is the height of the overhang to the nearest tenth of a meter?

1. Determine if the triangles are similar.

\[ \triangle ABC \] is a right triangle with \( \angle C \) the right angle.
\n\( \overline{CD} \) is the altitude of \( \triangle ABC \), creating two similar triangles, \( \triangle ACD \) and \( \triangle CBD \).

\[ \triangle ABC \sim \triangle ACD \sim \triangle CBD \]
2. Solve the problem.

Similar triangles have proportional sides.

Create a proportion to find the height of the overhang.

\[
\frac{AD}{CD} = \frac{CD}{BD}
\]

Create a proportion.

\[
\frac{1.75}{8.5} = \frac{8.5}{x}
\]

Substitute known values.

\[(1.75)(x) = (8.5)(8.5)\]

Find the cross products.

\[1.75x = 72.25\]

Simplify.

\[x \approx 41.3\]

Solve for \(x\).

The length of \(BD\) is approximately 41.3 meters; however, the measure of the overhang is represented by \(AB\).

Find the length of \(AB\).

\[41.3 + 1.75 = 43.05\]

The height of the overhang is approximately 43.1 meters.
Problem-Based Task 1.7.4: Too Tall?

Parks directors routinely assess the health of the trees in recreation areas. If trees are found to be diseased, they are often treated. If trees become too weak, they are removed before they become a danger to people and structures. Gorge Park is a rectangular park measuring 400 feet by 200 feet and is enclosed by a fence. A diseased tree needing removal stands in the center of the park. Tree removers must avoid having the tree fall on the fence. If necessary, the tree can be trimmed prior to being cut down. The 6-foot-tall parks director measured the length of the shadow cast by the tree to be 147 feet and the length of his own shadow to be 9 feet. Does the tree’s trunk need to be trimmed prior to cutting it down to avoid damaging the fence?
Problem-Based Task 1.7.4: Too Tall?

Coaching

a. Draw triangles to represent the height of the parks director, the diseased tree, and their shadows.

b. Are the triangles in your diagram similar? Explain.

c. What are the characteristics of similar triangles?

d. Is it possible to determine the height of the diseased tree using the given information? Explain your answer.

e. What is the height of the diseased tree?

f. Draw and label a diagram of Gorge Park and mark the location of the diseased tree.

g. Compare the height of the tree to the distance between the tree and each fence.

h. Does the tree need to be trimmed prior to cutting it down to avoid damaging the fence?
Problem-Based Task 1.7.4: Too Tall?

Coaching Sample Responses

a. Draw triangles to represent the height of the parks director, the diseased tree, and their shadows.

b. Are the triangles in your diagram similar? Explain.

The triangles in the diagram are similar because of the Angle-Angle (AA) Similarity Statement, which states that if two angles of one triangle are congruent to two angles of another triangle, then the triangles are similar.

c. What are the characteristics of similar triangles?

The angles of similar triangles are congruent and the sides are proportional.

d. Is it possible to determine the height of the diseased tree using the given information? Explain your answer.

Yes, it is possible to determine the height of the diseased tree using the given information by creating a proportion using the two similar triangles.
e. What is the height of the diseased tree?

Create a proportion to find the height of the diseased tree.

\[ \frac{\text{height of the director}}{\text{height of the tree}} = \frac{\text{length of the director's shadow}}{\text{length of the tree's shadow}} \]

\[ \frac{6}{x} = \frac{9}{147} \]

Substitute known values.

\[(6)(147) = (9)(x)\]

Find the cross products.

\[882 = 9x\]

Simplify.

\[x = 98\]

Solve for \(x\).

The height of the diseased tree is 98 feet.

f. Draw and label a diagram of Gorge Park and mark the location of the diseased tree.
g. Compare the height of the tree to the distance between the tree and each fence.

The diseased tree is located in the center of the park. According to the diagram, there are 200 feet of park to the east and west of the tree and 100 feet of park north and south of the tree. The height of the tree is less than both of these distances to the fence.

h. Does the tree need to be trimmed prior to cutting it down to avoid damaging the fence?

The tree does not need to be trimmed. If cut properly, the tree will not fall on the fence.

**Recommended Closure Activity**

Select one or more of the essential questions for a class discussion or as a journal entry prompt.
Practice 1.7.4: Solving Problems Using Similarity and Congruence

Use what you have learned about similar triangles to solve each problem.

1. A flat-roofed garage casts a shadow that is 9 meters long. At the same time, a 1.8-meter lamppost casts a shadow that is 2.7 meters long. What is the height of the garage?

2. A 12-foot statue casts a shadow that is 5 feet long. At the same time, a fence post casts a shadow that is 1.25 feet long. What is the height of the fence post?

For problems 3–10, use the information and the diagrams to solve each problem.

3. A piece of decorative trim is added to an asymmetrical roofline. What is the length of the decorative trim, $DE$?
4. A right-of-way parallel to Murch Road is to be constructed on a triangular plot of land. What is the length of the plot of land along Main Street between Murch Road and the right-of-way?

![Diagram of triangle with sides labeled 300 ft, 165 ft, and 220 ft.]

5. To measure $\overline{BC}$, the distance across a lake, a surveyor stands at point $A$ and locates points $B, C, D,$ and $E$. What is the distance across the lake?

![Diagram of lake with points $A, B, C, D,$ and $E$ and distances labeled.]

---

continued
6. To measure $\overline{BC}$, the distance across a crater, an archeologist stands at point $A$ and locates points $B, C, D,$ and $E$. What is the distance across the crater?

7. To estimate the height of his school, a student positions himself so that his line of sight to the top of the school and his line of sight to the bottom form a right angle. What is the height of the school?
8. To estimate the height of a monument, Cala positions herself so that her line of sight to the top of the monument and her line of sight to the bottom form a right angle. What is the height of the monument?

![Diagram of a right triangle with sides 5 ft, 12 ft, and hypotenuse x.]

9. The height of a ramp at a point 2.5 meters from its bottom edge is 1.2 meters. If the ramp runs for 6.7 meters along the ground, what is its height at its highest point, to the nearest tenth of a meter?

![Diagram of a right triangle with sides 2.5 m, 6.7 m, and 1.2 m, and hypotenuse x.]

continued
10. A geographer completed the following diagram to map a canyon’s width. Determine \( AR \), the distance across the canyon.
Lesson 8: Proving Theorems About Lines and Angles

Common Core Georgia Performance Standard
MCC9–12.G.CO.9

Essential Questions
1. How do angle relationships work together in two pairs of intersecting, opposite rays?
2. How do angle relationships work together in a set of parallel lines intersected by a transversal?
3. How are angle relationships important in the real world?
4. How do proofs apply to situations outside of mathematics?

WORDS TO KNOW
adjacent angles \hspace{1cm} \text{angles that lie in the same plane and share a vertex and a common side. They have no common interior points.}

alternate exterior angles \hspace{1cm} \text{angles that are on opposite sides of the transversal and lie on the exterior of the two lines that the transversal intersects}

alternate interior angles \hspace{1cm} \text{angles that are on opposite sides of the transversal and lie within the interior of the two lines that the transversal intersects}

complementary angles \hspace{1cm} \text{two angles whose sum is } 90^\circ

corresponding angles \hspace{1cm} \text{angles in the same relative position with respect to the transversal and the intersecting lines}

equidistant \hspace{1cm} \text{the same distance from a reference point}

exterior angles \hspace{1cm} \text{angles that lie outside a pair of parallel lines}

interior angles \hspace{1cm} \text{angles that lie between a pair of parallel lines}

linear pair \hspace{1cm} \text{a pair of adjacent angles whose non-shared sides form a straight angle}

nonadjacent angles \hspace{1cm} \text{angles that have no common vertex or common side, or have shared interior points}

perpendicular bisector \hspace{1cm} \text{a line that intersects a segment at its midpoint at a right angle}
### Instruction

- **perpendicular lines**: Two lines that intersect at a right angle (90°). The lines form four adjacent and congruent right angles.

- **plane**: A flat, two-dimensional figure without depth that has at least three non-collinear points and extends infinitely in all directions.

- **postulate**: A true statement that does not require a proof.

- **proof**: A set of justified statements organized to form a convincing argument that a given statement is true.

- **right angle**: An angle measuring 90°.

- **same-side exterior angles**: Angles that lie on the same side of the transversal and are outside the lines that the transversal intersects; sometimes called consecutive exterior angles.

- **same-side interior angles**: Angles that lie on the same side of the transversal and are in between the lines that the transversal intersects; sometimes called consecutive interior angles.

- **straight angle**: An angle with rays in opposite directions; i.e., a straight line.

- **supplementary angles**: Two angles whose sum is 180°.

- **transversal**: A line that intersects a system of two or more lines.

- **vertical angles**: Nonadjacent angles formed by two pairs of opposite rays.
Recommended Resources

- Interactivate. “Angles.”
  
  http://www.walch.com/rr/00027
  
  This website generates a set of parallel lines intersected by a transversal and prompts users to identify the angle relationships. The site provides immediate feedback.

  
  http://www.walch.com/rr/00028
  
  This online quiz allows users to receive immediate feedback about their answers. If the answer is incorrect, the program explains how to solve the problem correctly. This site deals with complementary, supplementary, vertical, adjacent, and congruent angles.

  
  http://www.walch.com/rr/00029
  
  This online quiz deals with angle relationships in a set of parallel lines intersected by a transversal, and provides immediate feedback. For incorrect answers, the program explains how to solve the problem correctly.

  
  http://www.walch.com/rr/00030
  
  This website gives a brief explanation of vertical angles and provides a manipulative so users can investigate how angle measures change as the positions of the intersecting lines change. There is a short quiz at the end of the lesson, as well as links to other sites on angle relationships.

  
  http://www.walch.com/rr/00031
  
  This online activity is provided in a game-show format that allows users to compete in teams or individually. The multiple-choice quiz questions pertain to angle relationships, and are worth points according to difficulty. Players receive immediate feedback.
Metalbro is a construction company involved with building a new skyscraper in Dubai. The diagram below is a rough sketch of a crane that Metalbro workers are using to build the skyscraper. The vertical line represents the support tower and the other line represents the boom. For safety reasons, the boom cannot be more than 15° beyond the horizontal in either direction. A horizontal line forms a 90° angle with the support tower. A straight line forms a 180° angle.

1. What are the safety requirements for $m\angle 1$?
2. What are the safety requirements for $m\angle 2$?
3. What are the safety requirements for $m\angle 3$?
4. What are the safety requirements for $m\angle 4$?
5. Use your findings to fill in the table.

<table>
<thead>
<tr>
<th>Based on lower boundary of $\angle 1$</th>
<th>Based on upper boundary of $\angle 1$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$m\angle 1$</td>
<td></td>
</tr>
<tr>
<td>$m\angle 2$</td>
<td></td>
</tr>
<tr>
<td>$m\angle 3$</td>
<td></td>
</tr>
<tr>
<td>$m\angle 4$</td>
<td></td>
</tr>
</tbody>
</table>

6. What do you notice about these angles?
Lesson 1.8.1: Proving the Vertical Angles Theorem

Common Core Georgia Performance Standard
MCC9–12.G.CO.9

Warm-Up 1.8.1 Debrief

Metalbro is a construction company involved with building a new skyscraper in Dubai. The diagram below is a rough sketch of a crane that Metalbro workers are using to build the skyscraper. The vertical line represents the support tower and the other line represents the boom. For safety reasons, the boom cannot be more than 15° beyond the horizontal in either direction. A horizontal line forms a 90° angle with the support tower. A straight line forms a 180° angle.

1. What are the safety requirements for $m\angle 1$?

   The boom cannot be more than 15° beyond the horizontal in either direction. Since a horizontal line forms a 90° angle with a vertical line, add 15° to 90° to find the upper boundary of the boom angle, and then subtract 15° from 90° to find the lower boundary of the boom angle.

   Upper boundary: $90 + 15 = 105$
   Lower boundary: $90 – 15 = 75$

   The safety requirements for $m\angle 1$ are that the angle must be between 75° and 105°.
2. What are the safety requirements for \( m\angle 2 \)?

\( \angle 1 \) and \( \angle 2 \) form a straight line, so the sum of their angles is 180º.

To find \( m\angle 2 \) when \( m\angle 1 = 75 \), set up an equation.

\[
\begin{align*}
m\angle 1 + m\angle 2 &= 180 \\
75 + m\angle 2 &= 180 \\
m\angle 2 &= 180 - 75 \\
m\angle 2 &= 105
\end{align*}
\]

To find \( m\angle 2 \) when \( m\angle 1 = 105 \), set up an equation.

\[
\begin{align*}
m\angle 1 + m\angle 2 &= 180 \\
105 + m\angle 2 &= 180 \\
m\angle 2 &= 180 - 105 \\
m\angle 2 &= 75
\end{align*}
\]

You could also see by inspection of the previous equation that 75 + 105 = 180.

When \( m\angle 1 = 75 \), \( m\angle 2 = 105 \), and when \( m\angle 1 = 105 \), \( m\angle 2 = 75 \).

The safety requirements for \( m\angle 2 \) are that the angle must be between 75º and 105º.

3. What are the safety requirements for \( m\angle 3 \)?

\( \angle 2 \) and \( \angle 3 \) form a straight line, so the sum of their angles is 180º.

To find \( m\angle 3 \) when \( m\angle 2 = 105 \), set up an equation or use inspection from the last problem.

\[
\begin{align*}
m\angle 2 + m\angle 3 &= 180 \\
105 + m\angle 3 &= 180 \\
m\angle 3 &= 180 - 105 \\
m\angle 3 &= 75
\end{align*}
\]

To find \( m\angle 3 \) when \( m\angle 2 = 75 \), set up an equation or use inspection.

\[
\begin{align*}
m\angle 2 + m\angle 3 &= 180 \\
75 + m\angle 3 &= 180 \\
m\angle 3 &= 180 - 75 \\
m\angle 3 &= 105
\end{align*}
\]

When \( m\angle 2 = 105 \), \( m\angle 3 = 75 \), and when \( m\angle 2 = 75 \), \( m\angle 3 = 105 \).

The safety requirements for \( m\angle 3 \) are that the angle must be between 75º and 105º.
4. What are the safety requirements for $m \angle 4$?

$\angle 3$ and $\angle 4$ form a straight line, so the sum of their angles is $180^\circ$.

To find $m \angle 4$ when $m \angle 3 = 75^\circ$, set up an equation or use inspection.

$m \angle 3 + m \angle 4 = 180$
$75 + m \angle 4 = 180$
$m \angle 4 = 180 - 75$
$m \angle 4 = 105$

To find $m \angle 4$ when $m \angle 3 = 105^\circ$, set up an equation or use inspection.

$m \angle 3 + m \angle 4 = 180$
$105 + m \angle 4 = 180$
$m \angle 4 = 180 - 105$
$m \angle 4 = 75$

The safety requirements for $m \angle 4$ are that the angle must be between $75^\circ$ and $105^\circ$.

5. Use your findings to fill in the table.

<table>
<thead>
<tr>
<th>Based on lower boundary of $\angle 1$</th>
<th>Based on upper boundary of $\angle 1$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$m \angle 1$</td>
<td>$75^\circ$</td>
</tr>
<tr>
<td>$m \angle 2$</td>
<td>$105^\circ$</td>
</tr>
<tr>
<td>$m \angle 3$</td>
<td>$75^\circ$</td>
</tr>
<tr>
<td>$m \angle 4$</td>
<td>$105^\circ$</td>
</tr>
</tbody>
</table>

6. What do you notice about these angles?

The angles that aren’t next to each other but are opposite the point of intersection are congruent. $\angle 1$ and $\angle 3$ are congruent. $\angle 2$ and $\angle 4$ are congruent. If the angles are next to each other and form a line, they add up to $180^\circ$.

**Connection to the Lesson**

- Students will learn the names for these angle relationships.
- Students will use these angle relationships for solving problems and creating proofs.
Lesson 8: Proving Theorems About Lines and Angles

Prerequisite Skills
This lesson requires the use of the following skills:

- identifying and labeling points, lines, and angles
- using the addition and subtraction properties of angles

Introduction
Think about crossing a pair of chopsticks and the angles that are created when they are opened at various positions. How many angles are formed? What are the relationships among those angles? This lesson explores angle relationships. We will be examining the relationships of angles that lie in the same plane. A plane is a two-dimensional figure, meaning it is a flat surface, and it extends infinitely in all directions. Planes require at least three non-collinear points. Planes are named using those points or a capital script letter. Since they are flat, planes have no depth.

Key Concepts
- Angles can be labeled with one point at the vertex, three points with the vertex point in the middle, or with numbers. See the examples that follow.

- Be careful when using one vertex point to name the angle, as this can lead to confusion.
- If the vertex point serves as the vertex for more than one angle, three points or a number must be used to name the angle.
• **Straight angles** are angles with rays in opposite directions—in other words, straight angles are straight lines.

<table>
<thead>
<tr>
<th>Straight angle</th>
<th>Not a straight angle</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\angle BCD) is a straight angle. Points (B), (C), and (D) lie on the same line.</td>
<td>(\angle PQR) is not a straight angle. Points (P), (Q), and (R) do not lie on the same line.</td>
</tr>
</tbody>
</table>

• **Adjacent angles** are angles that lie in the same plane and share a vertex and a common side. They have no common interior points.

• **Nonadjacent angles** have no common vertex or common side, or have shared interior points.

<table>
<thead>
<tr>
<th>Adjacent angles</th>
<th>Nonadjacent angles</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\angle ABC) is adjacent to (\angle CBD). They share vertex (B) and (BC). (\angle ABC) and (\angle CBD) have no common interior points.</td>
<td>(\angle ABE) is not adjacent to (\angle FCD). They do not have a common vertex. (\angle PQS) is not adjacent to (\angle PQR). They share common interior points within (\angle PQS).</td>
</tr>
</tbody>
</table>
• **Linear pairs** are pairs of adjacent angles whose non-shared sides form a straight angle.

<table>
<thead>
<tr>
<th>Linear pair</th>
<th>Not a linear pair</th>
</tr>
</thead>
<tbody>
<tr>
<td><img src="image1.png" alt="Diagram of Linear Pair" /></td>
<td><img src="image2.png" alt="Diagram of Not a Linear Pair" /></td>
</tr>
</tbody>
</table>

\(\angle ABC\) and \(\angle CBD\) are a linear pair. They are adjacent angles with non-shared sides, creating a straight angle. 
\(\angle ABE\) and \(\angle FCD\) are not a linear pair. They are not adjacent angles.

• **Vertical angles** are nonadjacent angles formed by two pairs of opposite rays.

<table>
<thead>
<tr>
<th>Vertical angles</th>
<th>Not vertical angles</th>
</tr>
</thead>
<tbody>
<tr>
<td><img src="image3.png" alt="Diagram of Vertical Angles" /></td>
<td><img src="image4.png" alt="Diagram of Not Vertical Angles" /></td>
</tr>
</tbody>
</table>

\(\angle ABC\) and \(\angle EBD\) are vertical angles.
\(\angle ABC \cong \angle EBD\)
\(\angle ABE\) and \(\angle CBD\) are vertical angles.
\(\angle ABE \cong \angle CBD\)
\(\angle ABC\) and \(\angle EBD\) are not vertical angles. \(\overrightarrow{BC}\) and \(\overrightarrow{BD}\) are not opposite rays. They do not form one straight line.
Postulate

**Angle Addition Postulate**

If $D$ is in the interior of $\angle ABC$, then $m\angle ABD + m\angle DBC = m\angle ABC$.

If $m\angle ABD + m\angle DBC = m\angle ABC$, then $D$ is in the interior of $\angle ABC$.

- Informally, the Angle Addition Postulate means that the measure of the larger angle is made up of the sum of the two smaller angles inside it.
- **Supplementary angles** are two angles whose sum is $180^\circ$.
- Supplementary angles can form a linear pair or be nonadjacent.
- In the following diagram, the angles form a linear pair.

$$m\angle ABD + m\angle DBC = 180$$

- The next diagram shows a pair of supplementary angles that are nonadjacent.

$$m\angle PQR + m\angle TUV = 180$$

Theorem

**Supplement Theorem**

If two angles form a linear pair, then they are supplementary.
• Angles have the same congruence properties that segments do.

**Theorem**

Congruence of angles is reflexive, symmetric, and transitive.

- Reflexive Property: \( \angle 1 \cong \angle 1 \)
- Symmetric Property: If \( \angle 1 \cong \angle 2 \), then \( \angle 2 \cong \angle 1 \).
- Transitive Property: If \( \angle 1 \cong \angle 2 \) and \( \angle 2 \cong \angle 3 \), then \( \angle 1 \cong \angle 3 \).

**Theorem**

Angles supplementary to the same angle or to congruent angles are congruent.

If \( m\angle 1 + m\angle 2 = 180 \) and \( m\angle 2 + m\angle 3 = 180 \), then \( \angle 1 \cong \angle 3 \).

• Perpendicular lines form four adjacent and congruent right angles.

**Theorem**

If two congruent angles form a linear pair, then they are right angles.
If two angles are congruent and supplementary, then each angle is a right angle.

• The symbol for indicating perpendicular lines in a diagram is a box at one of the right angles, as shown below.

- The symbol for writing perpendicular lines is \( \perp \), and is read as “is perpendicular to.”
In the diagram, $\overline{SQ} \perp \overline{PR}$.

Rays and segments can also be perpendicular.

In a pair of perpendicular lines, rays, or segments, only one right angle box is needed to indicate perpendicular lines.

Remember that perpendicular bisectors are lines that intersect a segment at its midpoint at a right angle; they are perpendicular to the segment.

Any point along the perpendicular bisector is equidistant from the endpoints of the segment that it bisects.

**Theorem**

**Perpendicular Bisector Theorem**

If a point lies on the perpendicular bisector of a segment, then that point is equidistant from the endpoints of the segment.

If a point is equidistant from the endpoints of a segment, then the point lies on the perpendicular bisector of the segment.

If $\overline{DE}$ is the perpendicular bisector of $\overline{AC}$, then $DA = DC$.

If $DA = DC$, then $\overline{DE}$ is the perpendicular bisector of $\overline{AC}$.

**Complementary angles** are two angles whose sum is 90°.

Complementary angles can form a right angle or be nonadjacent.

The following diagram shows a pair of nonadjacent complementary angles.
Theorem

Complement Theorem

If the non-shared sides of two adjacent angles form a right angle, then the angles are complementary.

Angles complementary to the same angle or to congruent angles are congruent.

Common Errors/Misconceptions

- not recognizing the theorem that is being used or that needs to be used
- setting expressions equal to each other rather than using the Complement or Supplement Theorems
- mislabeling angles with a single letter when that letter is the vertex for adjacent angles
- not recognizing adjacent and nonadjacent angles
Guided Practice 1.8.1

Example 1

Look at the following diagram. List pairs of supplementary angles, pairs of vertical angles, and a pair of opposite rays.

1. List pairs of supplementary angles.
   Supplementary angles have a sum of 180º.
   - \( \angle 5 \) and \( \angle 6 \) are adjacent supplementary angles. They form a linear pair.
   - \( \angle 1 \) and \( \angle 4 \) are adjacent supplementary angles. They form a linear pair.
   - \( \angle 2 \) and \( \angle 3 \) are adjacent supplementary angles. They form a linear pair.
   - \( \angle 7 \) and \( \angle 8 \) are adjacent supplementary angles. They form a linear pair.
   - \( \angle 1 \) and \( \angle 2 \) are adjacent supplementary angles. They form a linear pair.
   - \( \angle 3 \) and \( \angle 4 \) are adjacent supplementary angles. They form a linear pair.
Lesson 8: Proving Theorems About Lines and Angles

2. List the vertical angles.
   Vertical angles are nonadjacent angles that are formed by a pair of intersecting lines.

   \( \angle 1 \) and \( \angle 3 \) are vertical angles. They are formed by the intersecting segments of \( \overleftrightarrow{AC} \) and \( \overleftrightarrow{DF} \).
   \( \angle 2 \) and \( \angle 4 \) are vertical angles. They are formed by the intersecting segments of \( \overleftrightarrow{AC} \) and \( \overleftrightarrow{DF} \).

3. List a pair of opposite rays.
   Opposite rays form a straight angle.

   \( \overrightarrow{BA} \) and \( \overrightarrow{BC} \) are opposite rays.
   Also, \( \overrightarrow{EF} \) and \( \overrightarrow{ED} \) are opposite rays. This can be misleading since what is pictured represents segments, but remember that segments are just parts of lines and the line extends in both directions infinitely. From the line, any number of rays can be named.
Example 2

Prove the theorem that angles complementary to congruent angles are congruent using the given information.

In the figure below, prove that $\angle 1$ is congruent to $\angle 4$, given that $\overrightarrow{AC}$ is perpendicular to $\overrightarrow{CD}$ and $\angle 2$ is congruent to $\angle 3$.

1. Start by writing the given statements and the proof statement.
   
   Given: $\overrightarrow{AC} \perp \overrightarrow{CD}$; $\angle 2 \cong \angle 3$
   
   Prove: $\angle 1 \cong \angle 4$

2. Start writing what you know about perpendicular lines and complementary angles.
   
   Perpendicular lines form four adjacent and congruent right angles.
   Complementary angles have a sum of 90°, which is a right angle.
3. Determine where the right angles are located and use the Complement Theorem.

\( \angle ACD \) is a right angle because of the given information that \( \overrightarrow{AC} \perp \overrightarrow{CD} \). \( \angle ACD \) is made up of two adjacent complementary angles, \( \angle 1 \) and \( \angle 2 \). Therefore, \( m\angle 1 + m\angle 2 = 90 \).

\( \angle BCD \) is a right angle because of the given information that \( \overrightarrow{AC} \perp \overrightarrow{CD} \). \( \angle BCD \) is made up of two adjacent complementary angles, \( \angle 3 \) and \( \angle 4 \). Therefore, \( m\angle 3 + m\angle 4 = 90 \).

4. Use the definition of congruence.

Since \( \angle 2 \cong \angle 3 \), \( m\angle 2 = m\angle 3 \). The definition of congruence states that if two angles are congruent, then the measures of their angles are equal.

5. Use substitution.

Since \( m\angle 1 + m\angle 2 = 90 \) and \( m\angle 2 = m\angle 3 \), \( m\angle 1 + m\angle 3 = 90 \). Notice that \( m\angle 3 \) was substituted in for \( m\angle 2 \).

Also, as stated in step 3, \( m\angle 3 + m\angle 4 = 90 \). Since two expressions (\( m\angle 1 + m\angle 3 \) and \( m\angle 3 + m\angle 4 \)) both equal 90, set those two expressions equal to each other.

\[ m\angle 1 + m\angle 3 = m\angle 3 + m\angle 4 \]

6. Use the Reflexive Property.

\[ m\angle 3 = m\angle 3 \]
7. Use the Subtraction Property.

\[ m\angle 1 + m\angle 3 = m\angle 3 + m\angle 4 \]
Set the expressions equal to each other.

\[ m\angle 1 = m\angle 4 \]
Subtract \( m\angle 3 \) from both sides of the equation.

8. Use the definition of congruent angles.

\[ m\angle 1 = m\angle 4 \]
\[ \angle 1 \cong \angle 4 \]

9. Organize the information into a paragraph proof.

From the given information, \( \overrightarrow{AC} \) is perpendicular to \( \overrightarrow{CD} \). By the definition of perpendicular lines, these perpendicular lines create four right angles. Two of the right angles are \( \angle ACD \) and \( \angle BCD \). Each of the angles is made up of two smaller angles. By the Complement Theorem, \( m\angle 1 + m\angle 2 = 90 \) and \( m\angle 3 + m\angle 4 = 90 \). Since \( \angle 2 \cong \angle 3 \), the measures of \( \angle 2 \) and \( \angle 3 \) are equal according to the definition of congruence. \( m\angle 3 \) can be substituted into the first complementary angle equation for \( m\angle 2 \) so that \( m\angle 1 + m\angle 3 = 90 \). Since two expressions are set equal to 90, they are equal to each other; therefore, \( m\angle 1 + m\angle 3 = m\angle 3 + m\angle 4 \). Congruence of angles is reflexive, meaning that \( m\angle 3 = m\angle 3 \). This angle can be subtracted from both sides of the equation, leaving \( m\angle 1 = m\angle 4 \). By the definition of congruent angles, \( \angle 1 \cong \angle 4 \).
Example 3

In the diagram below, $\overline{AC}$ and $\overline{BD}$ are intersecting lines. If $m\angle 1 = 3x + 14$ and $m\angle 2 = 9x + 22$, find $m\angle 3$ and $m\angle 4$.

1. Use the Supplement Theorem.
   Since $\overline{BD}$ is a straight line, $m\angle 1 + m\angle 2 = 180$.

2. Use substitution to find the value of $x$.
   Substitute the measures of $\angle 1$ and $\angle 2$ into the equation $m\angle 1 + m\angle 2 = 180$.
   
   $m\angle 1 = 3x + 14$
   $m\angle 2 = 9x + 22$
   
   $m\angle 1 + m\angle 2 = 180$   Supplement Theorem
   $(3x + 14) + (9x + 22) = 180$   Substitute $3x + 14$ and $9x + 22$ for $m\angle 1$ and $m\angle 2$.
   $12x + 36 = 180$   Combine like terms.
   $12x = 144$   Subtract 36 from both sides.
   $x = 12$   Divide both sides by 12.
3. Use substitution to find $m\angle 1$.

   $m\angle 1 = 3x + 14$ and $x = 12$  
   $m\angle 1 = 3(12) + 14$  
   $m\angle 1 = 36 + 14$  
   $m\angle 1 = 50$

   Substitute 12 for $x$.

   Multiply.

   Add.

4. Use substitution to find $m\angle 2$.

   $m\angle 2 = 9x + 22$ and $x = 12$  
   $m\angle 2 = 9(12) + 22$  
   $m\angle 2 = 108 + 22$  
   $m\angle 2 = 130$

   Substitute 12 for $x$.

   Multiply.

   Add.

5. Use the Vertical Angles Theorem to find $m\angle 3$ and $m\angle 4$.

   $\angle 1$ and $\angle 3$ are vertical angles.  
   $\angle 1 \equiv \angle 3$  
   $m\angle 1 = m\angle 3$  
   $50 = m\angle 3$  
   Substitute 50 for $m\angle 1$.

   $\angle 2$ and $\angle 4$ are vertical angles.  
   $\angle 2 \equiv \angle 4$  
   $m\angle 2 = m\angle 4$  
   $130 = m\angle 4$  
   Substitute 130 for $m\angle 2$.

   $m\angle 3 = 50; m\angle 4 = 130$

The measure of $\angle 3$ is $50^\circ$ and the measure of $\angle 4$ is $130^\circ$.  


Example 4

Prove that vertical angles are congruent given a pair of intersecting lines, $\overline{AC}$ and $\overline{BD}$.

1. Draw a diagram and label three adjacent angles.

   ![Diagram](image)

2. Start with the Supplement Theorem.
   Supplementary angles add up to $180^\circ$.
   
   $m\angle 1 + m\angle 2 = 180$
   
   $m\angle 2 + m\angle 3 = 180$

3. Use substitution.
   Both expressions are equal to 180, so they are equal to each other.
   Rewrite the first equation, substituting $m\angle 2 + m\angle 3$ in for 180.
   
   $m\angle 1 + m\angle 2 = m\angle 2 + m\angle 3$

4. Use the Reflexive Property.
   $m\angle 2 = m\angle 2$
5. Use the Subtraction Property.
   Since $m\angle 2 = m\angle 2$, these measures can be subtracted out of the equation $m\angle 1 + m\angle 2 = m\angle 2 + m\angle 3$.
   This leaves $m\angle 1 = m\angle 3$.

6. Use the definition of congruence.
   Since $m\angle 1 = m\angle 3$, by the definition of congruence, $\angle 1 \cong \angle 3$.
   $\angle 1$ and $\angle 3$ are vertical angles and they are congruent. This proof also shows that angles supplementary to the same angle are congruent.

Example 5
In the diagram below, $\overline{DB}$ is the perpendicular bisector of $\overline{AC}$. If $AD = 4x - 1$ and $DC = x + 11$, what are the values of $AD$ and $DC$?

1. Use the Perpendicular Bisector Theorem to determine the values of $AD$ and $DC$.
   If a point is on the perpendicular bisector of a segment, then that point is equidistant from the endpoints of the segment being bisected. That means $AD = DC$. 
2. Use substitution to solve for $x$.

Given equations

$AD = 4x - 1$ and $DC = x + 11$

$AD = DC$ Perpendicular Bisector Theorem

$4x - 1 = x + 11$ Substitute $4x - 1$ for $AD$ and $x + 11$ for $DC$.

$3x = 12$ Combine like terms.

$x = 4$ Divide both sides of the equation by 3.

3. Substitute the value of $x$ into the given equations to determine the values of $AD$ and $DC$.

$AD = 4x - 1$ $DC = x + 11$

$AD = 4(4) - 1$ $DC = (4) + 11$

$AD = 15$ $DC = 15$

$AD$ and $DC$ are each 15 units long.
Problem-Based Task 1.8.1: Cutting Kitchen Tiles

Maresol is retiling the backsplash over her kitchen stove, and has to cut square ceramic tiles into 4 congruent triangles to create the pattern she wants. If she doesn’t cut each square into perfectly equal triangles, the tiles won’t fit together properly. Before she cuts the first tile, she uses a pencil to draw two segments on the tile. The segments form perpendicular bisectors. Use what you know about triangle congruency and perpendicular bisectors to prove that the 4 triangles are congruent.
Problem-Based Task 1.8.1: Cutting Kitchen Tiles

Coaching

a. What information are you given?

b. What is the definition of a perpendicular bisector?

c. Use the definition of a perpendicular bisector to determine which segments are congruent. Mark them in your diagram.

d. What does the Perpendicular Bisector Theorem state?

e. Write a statement of congruence using the Perpendicular Bisector Theorem.

f. Go to the next consecutive vertex and write a second statement of congruence using the Perpendicular Bisector Theorem.

g. Go to the next consecutive vertex and write a third statement of congruence using the Perpendicular Bisector Theorem.

h. Go to the next consecutive vertex and write a fourth statement of congruence using the Perpendicular Bisector Theorem.

i. On the diagram, mark the congruent sides that you developed in parts e–h.

j. What criteria do you need to prove triangles are congruent?

k. Are the four sections of the tile congruent?
Problem-Based Task 1.8.1: Cutting Kitchen Tiles

Coaching Sample Responses

a. What information are you given?

AC and DB are perpendicular bisectors of each other.

b. What is the definition of a perpendicular bisector?

A perpendicular bisector is a line that intersects a segment at a right angle at the midpoint of the segment.

c. Use the definition of a perpendicular bisector to determine which segments are congruent. Mark them in your diagram.

Since AC is the perpendicular bisector of DB, then DE ≅ EB.

Since DB is the perpendicular bisector of AC, then AE ≅ EC.


d. What does the Perpendicular Bisector Theorem state?

Any point on a perpendicular bisector is equidistant from the endpoints of the segment it bisects.
e. Write a statement of congruence using the Perpendicular Bisector Theorem.
   Since point $A$ is on the perpendicular bisector of $DB$, $AD \cong AB$.

f. Go to the next consecutive vertex and write a second statement of congruence using the Perpendicular Bisector Theorem.
   Since point $B$ is on the perpendicular bisector of $AC$, $BA \cong BC$.

g. Go to the next consecutive vertex and write a third statement of congruence using the Perpendicular Bisector Theorem.
   Since point $C$ is on the perpendicular bisector of $DB$, $CB \cong CD$.

h. Go to the next consecutive vertex and write a fourth statement of congruence using the Perpendicular Bisector Theorem.
   Since point $D$ is on the perpendicular bisector of $AC$, $DC \cong DA$.

i. On the diagram, mark the congruent sides that you developed in parts e–h.
j. What criteria do you need to prove triangles are congruent?
   Side-side-side (SSS), side-angle-side (SAS), or angle-side-angle (ASA).

k. Are the four sections of the tile congruent?
   Yes, the four triangles are congruent. $\triangle ABE \cong \triangle CBE \cong \triangle CDE \cong \triangle ADE$ by side-side-side (SSS).

**Recommended Closure Activity**
Select one or more of the essential questions for a class discussion or as a journal entry prompt.
1. List two pairs of adjacent angles and two pairs of nonadjacent angles.

2. List two pairs of supplementary angles. Write a statement about those angles using the Supplement Theorem.

3. List a pair of vertical angles. Write a statement about those angles using the Vertical Angles Theorem.

4. List a pair of complementary angles. Write a statement about those angles using the Complement Theorem.
In the diagram that follows, \( \overrightarrow{AC} \) and \( \overrightarrow{BD} \) intersect. Use this information to solve for the measures of the unknown angles in problems 5 and 6. Show and justify your work.

5. Find \( m\angle 4 \) if \( m\angle 1 = 3x + 4 \) and \( m\angle 2 = 2x - 4 \).

6. Find \( m\angle 1 \) if \( m\angle 1 = 13x + 7 \) and \( m\angle 3 = 7x + 49 \).
Use the diagram that follows to solve problems 7 and 8.

7. Find $m\angle 1$ given the following: $\angle 3$ and $\angle 4$ are complementary, $\angle 2 \equiv \angle 3$, $m\angle 2 = 6x + 24$, and $m\angle 4 = 5x$.

8. Find $m\angle 1$ if $\angle 2$ and $\angle 3$ are complementary, $m\angle 1 = 4x - 23$, and $m\angle 4 = x + 38$.
9. Prove that vertical angles are congruent.

10. Prove that a point on a perpendicular bisector is equidistant from the endpoints of the segment it bisects given that in $\triangle ACD$, $BD$ is the perpendicular bisector of $AC$ and point $E$ is on $BD$. Write your answer in a proof of your choosing.

   Given: $DB$ is the perpendicular bisector of $AC$.
   
   $E$ is a point on $DB$.
   
   Prove: $EA = EC$
City planners use geometry when building roads. Below is a portion of a city street map. In the diagram, \( \triangle BAE \sim \triangle CAF \sim \triangle DAG \). Use what you know about similar triangles and angle relationships to answer the questions that follow.

1. If \( m \angle 8 = 30 \) and \( m \angle 7 = 80 \), find \( m \angle 2 \). Justify your reasoning.

2. Using the angle measures from problem 1, find the rest of the angle measures and state what angle relationship you used to find each angle measure. Use the following table to help organize the information.

<table>
<thead>
<tr>
<th>Angle</th>
<th>Measure</th>
<th>Angle relationship used to determine measure</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td></td>
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<td>3</td>
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<td>5</td>
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<td>6</td>
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<td>7</td>
<td></td>
<td></td>
</tr>
<tr>
<td>8</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
1. If $m \angle 8 = 30^\circ$ and $m \angle 7 = 80^\circ$, find $m \angle 2$. Justify your reasoning.

The sum of the measures of the interior angles is $180^\circ$. Sum the measures of the two given angles and subtract that from $180^\circ$.

$m \angle 2 + m \angle 7 + m \angle 8 = 180$
$m \angle 2 + 80 + 30 = 180$
$m \angle 2 = 180 - 110$
$m \angle 2 = 70$

The measure of $\angle 2$ is $70^\circ$. 
2. Using the angle measures from problem 1, find the rest of the angle measures and state what angle relationship you used to find the angle measure. Use the following table to help organize the information.

<table>
<thead>
<tr>
<th>Angle</th>
<th>Measure</th>
<th>Angle relationship used to determine measure</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>110°</td>
<td>Linear pairs are supplementary. ( \angle 1 ) and ( \angle 2 ) are a linear pair.</td>
</tr>
<tr>
<td>2</td>
<td>70°</td>
<td>The sum of the interior angles of a triangle equals 180°.</td>
</tr>
<tr>
<td>3</td>
<td>110°</td>
<td>Vertical angles are congruent. ( \angle 1 ) and ( \angle 3 ) are vertical angles.</td>
</tr>
<tr>
<td>4</td>
<td>80°</td>
<td>Corresponding angles of similar triangles are congruent. ( \angle 4 ) and ( \angle 7 ) are corresponding angles in similar triangles.</td>
</tr>
<tr>
<td>5</td>
<td>100°</td>
<td>Linear pairs are supplementary. ( \angle 4 ) and ( \angle 5 ) are a linear pair.</td>
</tr>
<tr>
<td>6</td>
<td>100°</td>
<td>Vertical angles are congruent. ( \angle 5 ) and ( \angle 6 ) are vertical angles.</td>
</tr>
<tr>
<td>7</td>
<td>80°</td>
<td>Given</td>
</tr>
<tr>
<td>8</td>
<td>30°</td>
<td>Given</td>
</tr>
</tbody>
</table>

Connection to the Lesson

- Students will use corresponding angles of parallel lines intersected by a transversal. This warm-up gives students a visual of where corresponding angles lie that relates back to their work with similar triangles.

- The warm-up gives students practice with finding supplementary angles given the value of one angle. This will be extended to finding supplementary angles where the angles are given as expressions.

- Students will extend their knowledge of similar triangles where the Triangle Proportionality Theorem was used to determine angle relationships formed by parallel lines intersected by a transversal.
Introduction

Think about all the angles formed by parallel lines intersected by a transversal. What are the relationships among those angles? In this lesson, we will prove those angle relationships. First, look at a diagram of a pair of parallel lines and notice the interior angles versus the exterior angles. The **interior angles** lie between the parallel lines and the **exterior angles** lie outside the pair of parallel lines. In the following diagram, line \( k \) is the transversal. A **transversal** is a line that intersects a system of two or more lines. Lines \( l \) and \( m \) are parallel. The exterior angles are \( \angle 1, \angle 2, \angle 7, \) and \( \angle 8 \). The interior angles are \( \angle 3, \angle 4, \angle 5, \) and \( \angle 6 \).
Key Concepts

- A straight line has a constant slope and parallel lines have the same slope.
- If a line crosses a set of parallel lines, then the angles in the same relative position have the same measures.
- Angles in the same relative position with respect to the transversal and the intersecting lines are corresponding angles.
- If the lines that the transversal intersects are parallel, then corresponding angles are congruent.

**Postulate**

**Corresponding Angles Postulate**

If two parallel lines are cut by a transversal, then corresponding angles are congruent.

Corresponding angles:

\[ \angle 1 \cong \angle 5, \quad \angle 2 \cong \angle 6, \quad \angle 3 \cong \angle 7, \quad \angle 4 \cong \angle 8 \]

The converse is also true. If corresponding angles of lines that are intersected by a transversal are congruent, then the lines are parallel.
• **Alternate interior angles** are angles that are on opposite sides of the transversal and lie on the interior of the two lines that the transversal intersects.

• If the two lines that the transversal intersects are parallel, then alternate interior angles are congruent.

### Theorem

**Alternate Interior Angles Theorem**

If two parallel lines are intersected by a transversal, then alternate interior angles are congruent.

Alternate interior angles:

\[ \angle 3 \cong \angle 6, \quad \angle 4 \cong \angle 5 \]

The converse is also true. If alternate interior angles of lines that are intersected by a transversal are congruent, then the lines are parallel.
• **Same-side interior angles** are angles that lie on the same side of the transversal and are in between the lines that the transversal intersects.

• If the lines that the transversal intersects are parallel, then same-side interior angles are supplementary.

• Same-side interior angles are sometimes called consecutive interior angles.

**Theorem**

**Same-Side Interior Angles Theorem**

If two parallel lines are intersected by a transversal, then same-side interior angles are supplementary.

```
<table>
<thead>
<tr>
<th>1</th>
<th>2</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>4</td>
</tr>
</tbody>
</table>
```

Same-side interior angles:

\[ m\angle 3 + m\angle 5 = 180 \]

\[ m\angle 4 + m\angle 6 = 180 \]

The converse is also true. If same-side interior angles of lines that are intersected by a transversal are supplementary, then the lines are parallel.
Alternate exterior angles are angles that are on opposite sides of the transversal and lie on the exterior (outside) of the two lines that the transversal intersects.

If the two lines that the transversal intersects are parallel, then alternate exterior angles are congruent.

**Theorem**

**Alternate Exterior Angles Theorem**

If parallel lines are intersected by a transversal, then alternate exterior angles are congruent.

Alternate exterior angles:

\[ \angle 1 \cong \angle 8, \quad \angle 2 \cong \angle 7 \]

The converse is also true. If alternate exterior angles of lines that are intersected by a transversal are congruent, then the lines are parallel.
• Same-side exterior angles are angles that lie on the same side of the transversal and are outside the lines that the transversal intersects.

• If the lines that the transversal intersects are parallel, then same-side exterior angles are supplementary.

• Same-side exterior angles are sometimes called consecutive exterior angles.

**Theorem**

**Same-Side Exterior Angles Theorem**

If two parallel lines are intersected by a transversal, then same-side exterior angles are supplementary.

$$m \angle 1 + m \angle 7 = 180$$

$$m \angle 2 + m \angle 8 = 180$$

The converse is also true. If same-side exterior angles of lines that are intersected by a transversal are supplementary, then the lines are parallel.
When the lines that the transversal intersects are parallel and perpendicular to the transversal, then all the interior and exterior angles are congruent right angles.

**Theorem**

**Perpendicular Transversal Theorem**

If a line is perpendicular to one line that is parallel to another, then the line is perpendicular to the second parallel line.

The converse is also true. If a line intersects two lines and is perpendicular to both lines, then the two lines are parallel.
Common Errors/Misconceptions

- setting expressions equal to each other instead of setting up expressions as a supplemental relationship and vice versa
- not being able to recognize the relative positions of the angles in a set of parallel lines intersected by a transversal
- misidentifying or not being able to identify the theorem or postulate to apply
- leaving out definitions or other steps in proofs
- assuming information not given in a diagram or problem statement that cannot be assumed
- assuming drawings are to scale
Guided Practice 1.8.2

Example 1

Given $\overline{AB} \parallel \overline{DE}$, prove that $\triangle ABC \sim \triangle DEC$.

1. State the given information.
   $\overline{AB} \parallel \overline{DE}$

2. Extend the lines in the figure to show the transversals.
   Indicate the corresponding angles and mark the congruence of the corresponding angles with arcs.

$\angle CAB \cong \angle CDE$ and $\angle CBA \cong \angle CED$ because each pair is a set of corresponding angles.
3. Use the AA (angle-angle) criteria.

When two pairs of corresponding angles of a triangle are congruent, the angles are similar.

In this case, we actually know that all three pairs of corresponding angles are congruent because \( \angle C \cong \angle C \) by the Reflexive Property.

4. Write the information in a two-column proof.

<table>
<thead>
<tr>
<th>Statements</th>
<th>Reasons</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. ( \overline{AB} \parallel \overline{DE} )</td>
<td>1. Given</td>
</tr>
<tr>
<td>2. ( \angle CAB \cong \angle CDE, \angle CBA \cong \angle CED )</td>
<td>2. Corresponding Angles Postulate</td>
</tr>
<tr>
<td>3. ( \triangle ABC \sim \triangle DEC )</td>
<td>3. AA Postulate</td>
</tr>
</tbody>
</table>
Example 2

Given two parallel lines and a transversal, prove that alternate interior angles are congruent. In the following diagram, lines $l$ and $m$ are parallel. Line $k$ is the transversal.

Given: $l \parallel m$, and line $k$ is a transversal.

Prove: $\angle 3 \cong \angle 6$

1. State the given information.
   
   $l \parallel m$, and line $k$ is a transversal.
2. Use the Corresponding Angles Postulate.

Corresponding angles are angles that lie in the same relative position with respect to the transversal and the lines the transversal intersects. If the lines that the transversal intersects are parallel, then corresponding angles are congruent. \( \angle 3 \) and \( \angle 7 \) are corresponding angles because they are both below the parallel lines and on the left side of the transversal.

\( \angle 3 \cong \angle 7 \) because they are corresponding angles.

3. Use the Vertical Angles Theorem.

Vertical angles are formed when a pair of lines intersect. Vertical angles are the nonadjacent angles formed by these intersecting lines. Vertical angles are congruent.

\( \angle 7 \cong \angle 6 \) because they are vertical angles.

4. Use the Transitive Property.

Since \( \angle 3 \cong \angle 7 \) and \( \angle 7 \cong \angle 6 \), \( \angle 3 \cong \angle 6 \).

5. Write the information in a flow proof.

```
Given: \( \ell \parallel m \)

\( \angle 3 \cong \angle 7 \)  
Corresponding Angles Postulate

\( \angle 7 \cong \angle 6 \)  
Vertical Angles Theorem

\( \angle 3 \cong \angle 6 \)  
Transitive Property
```

Line \( \ell \) is a transversal.
Example 3
In the following diagram, $\overline{AB} \parallel \overline{CD}$ and $\overline{AC} \parallel \overline{BD}$. If $m\angle 1 = 3(x + 15)$, $m\angle 2 = 2x + 55$, and $m\angle 3 = 4y + 9$, find the measures of the unknown angles and the values of $x$ and $y$.

1. Find the relationship between two angles that have the same variable. 
   $\angle 1$ and $\angle 2$ are same-side interior angles and are both expressed in terms of $x$.

2. Use the Same-Side Interior Angles Theorem. 
   Same-side interior angles are supplementary. Therefore, $m\angle 1 + m\angle 2 = 180$. 
3. Use substitution and solve for $x$.

$\angle 1 = 3(x+15)$ and $\angle 2 = 2x+55$

$\angle 1 + \angle 2 = 180$

$[3(x + 15)] + (2x + 55) = 180$

$(3x + 45) + (2x + 55) = 180$

$5x + 100 = 180$

$5x = 80$

$x = 16$

Given

Same-Side Interior Angles Theorem

Substitute $3(x + 15)$ for $\angle 1$ and $2x + 55$ for $\angle 2$.

Distribute.

Combine like terms.

Subtract 100 from both sides of the equation.

Divide both sides by 5.

4. Find $\angle 1$ and $\angle 2$ using substitution.

$\angle 1 = 3(x+15); x = 16$

$\angle 2 = 2x+55; x = 16$

$m\angle 1 = 3[(16)+15]$

$m\angle 1 = 3(31)$

$m\angle 1 = 93$

$m\angle 2 = 32+55$

$m\angle 2 = 87$

After finding $\angle 1$, to find $\angle 2$ you could alternately use the Same-Side Interior Angles Theorem, which says that same-side interior angles are supplementary.

$m\angle 1 + m\angle 2 = 180$

$(93) + m\angle 2 = 180$

$m\angle 2 = 180 - 93$

$m\angle 2 = 87$

5. Find the relationship between one of the known angles and the last unknown angle, $\angle 3$.

$\angle 1$ and $\angle 3$ lie on the opposite side of the transversal on the interior of the parallel lines. This means they are alternate interior angles.
6. Use the Alternate Interior Angles Theorem.

The Alternate Interior Angles Theorem states that alternate interior angles are congruent if the transversal intersects a set of parallel lines. Therefore, \( \angle 1 \cong \angle 3 \).

7. Use the definition of congruence and substitution to find \( m \angle 3 \).

\[ \angle 1 \cong \angle 3 \], so \( m \angle 1 = m \angle 3 \).

\[ m \angle 1 = 93 \]

Using substitution, \( 93 = m \angle 3 \).

8. Use substitution to solve for \( y \).

\[ m \angle 3 = 4y + 9 \quad \text{Given} \]
\[ 93 = 4y + 9 \quad \text{Substitute 93 for } m \angle 3. \]
\[ 84 = 4y \quad \text{Subtract 9 from both sides of the equation.} \]
\[ y = 21 \quad \text{Simplify.} \]
Example 4

In the following diagram, \( \overline{AB} \parallel \overline{CD} \). If \( m \angle 1 = 35 \) and \( m \angle 2 = 65 \), find \( m \angle EQF \).

1. Draw a third parallel line that passes through point \( Q \).
   Label a second point on the line as \( P \). \( \overline{PQ} \parallel \overline{AB} \parallel \overline{CD} \).
2. Use $\overrightarrow{QE}$ as a transversal to $\overrightarrow{AB}$ and $\overrightarrow{PQ}$ and identify angle relationships.

$\angle 1 \cong \angle BEQ$ because they are vertical angles.

$\angle BEQ \cong \angle EQP$ because they are alternate interior angles.

$\angle 1 \cong \angle EQP$ by the Transitive Property.

It was given that $m\angle 1 = 35$.

By substitution, $m\angle EQP = 35$.

3. Use $\overrightarrow{QF}$ as a transversal to $\overrightarrow{PQ}$ and $\overrightarrow{CD}$ and identify angle relationships.

$\angle 2 \cong \angle FQP$ because they are alternate interior angles.

It was given that $m\angle 2 = 65$.

By substitution, $m\angle FQP = 65$.

4. Use angle addition.

Notice that the angle measure we are looking for is made up of two smaller angle measures that we just found.

$m\angle EQF = m\angle EQP + m\angle FQP$

$m\angle EQF = 35 + 65$

$m\angle EQF = 100$
Problem-Based Task 1.8.2: Retinal Refraction

When a person looks at an object, the light rays are refracted or distorted as they pass through the eye, and the image is transmitted upside down on the retina in the back of the eye. The object and its retinal image are similar. Prove that they are in proportion using similar triangles. A diagram is given below. Assume that both the person looking at the image and the image are vertical.
Problem-Based Task 1.8.2: Retinal Refraction

Coaching

a. Label the angles on the diagram.

b. What information is given?

c. Which angles are congruent in parallel lines intersected by a transversal?

d. Which angles are congruent in the diagram and why?

e. What criteria are needed to prove that triangles are similar?

f. Which two triangles are similar?

g. What is the proof that the triangles are similar?

h. What is true about the parts of similar triangles?
Problem-Based Task 1.8.2: Retinal Refraction

Coaching Sample Responses

a. Label the angles on the diagram.

b. What information is given?
   \( \overline{AB} \) and \( \overline{DE} \) are both vertical. Therefore, \( \overline{AB} \parallel \overline{DE} \).

c. Which angles are congruent in parallel lines intersected by a transversal?
   Alternate interior angles are congruent and alternate exterior angles are congruent.
d. Which angles are congruent in the diagram and why?

\[ \angle CED \cong \angle BAE \] by the Alternate Interior Angles Theorem.

\[ \angle CDE \cong \angle ABD \] by the Alternate Interior Angles Theorem.

\[ \angle ACB \cong \angle DCE \] by the Vertical Angles Theorem.

e. What criteria are needed to prove that triangles are similar?
The needed criteria are two angles (AA), two sides and the included angle (SAS), or all three sides (SSS).

f. Which two triangles are similar?
\[ \triangle ABC \sim \triangle EDC \]

g. What is the proof that the triangles are similar?

<table>
<thead>
<tr>
<th>Statements</th>
<th>Reasons</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. ( AB ) and ( DE ) are both vertical.</td>
<td>1. Given</td>
</tr>
<tr>
<td>2. ( AB \parallel DE )</td>
<td>2. Definition of parallel lines</td>
</tr>
<tr>
<td>3. ( \angle CED \cong \angle BAE )</td>
<td>3. Alternate Interior Angles Theorem</td>
</tr>
<tr>
<td>4. ( \angle ACB \cong \angle DCE )</td>
<td>4. Vertical Angles Theorem</td>
</tr>
<tr>
<td>5. ( \triangle ABC \sim \triangle EDC )</td>
<td>5. Angle-Angle Similarity Postulate</td>
</tr>
</tbody>
</table>

h. What is true about the parts of similar triangles?
The corresponding parts of similar triangles are in proportion.

**Recommended Closure Activity**
Select one or more of the essential questions for a class discussion or as a journal entry prompt.
Practice 1.8.2: Proving Theorems About Angles in Parallel Lines Cut by a Transversal

Use the following diagram to solve problems 1–5, given that $AB \parallel CD$ and line $\ell$ is the transversal. Justify your answers using angle relationships in parallel lines intersected by a transversal.

1. Find $m\angle 5$ if $m\angle 5 = 2(3x + 13)$ and $m\angle 7 = 3x + 50$.

2. Find $m\angle 2$ if $m\angle 2 = 4x + 39$ and $m\angle 7 = 12x - 17$.

3. Find $m\angle 6$ if $m\angle 6 = 7x + 41$ and $m\angle 7 = 3x - 1$.

4. Find $m\angle 4$ if $m\angle 4 = 2(5x - 9)$ and $m\angle 5 = 3(x + 8)$.

5. Find $m\angle 1$ if $m\angle 1 = 11x + 35$ and $m\angle 4 = x + 1$. 

continued
Use the following diagram to solve problems 6 and 7. Given: $AB \parallel CD$.

6. Find $m \angle EQF$ if $m \angle 1 = 110$ and $m \angle 2 = 135$.

7. Find $m \angle EQF$ if $m \angle 1 = 117$ and $m \angle 3 = 31$. 

continued
Use the following diagram to solve problem 8. Given: \( \overline{AF} \parallel \overline{BE}, \overline{HC} \parallel \overline{GD} \), \( \angle 1 = 5x - 16 \), \( \angle 2 = 6x - 13 \), and \( \angle 3 = 10y - 9 \).

8. Find the measures of the numbered angles and the values of \( x \) and \( y \). Justify your reasoning.

Write proofs to complete problems 9 and 10.

9. Prove the Same-Side Interior Angles Theorem.

10. Prove that alternate exterior angles are congruent. Write your answer in a paragraph proof.
Lesson 9: Proving Theorems About Triangles

Common Core Georgia Performance Standard
MCC9–12.G.CO.10

Essential Questions
1. What is the relationship between an interior and exterior angle at the same vertex?
2. What is the relationship between an exterior angle and the remote interior angles of a triangle?
3. How are properties of isosceles triangles used and applied?
4. What relationships exist between the midsegment and a triangle?
5. How can the point of concurrency be used to solve problems?
6. What are the similarities and differences among medians, altitudes, perpendicular bisectors, and angle bisectors of a triangle?
7. Is there only one center of a circle?

WORDS TO KNOW

acute triangle: a triangle in which all of the angles are acute (less than 90°)

auxiliary line: a line added to a figure to help prove a statement

base: the side that is opposite the vertex angle of an isosceles triangle

base angle: an angle formed by the base and one congruent side of an isosceles triangle

centroid: the intersection of the medians of a triangle

circumcenter: the intersection of the perpendicular bisectors of a triangle

circumscribed circle: a circle that contains all vertices of a polygon

concurrent lines: lines that intersect at one point

coordinate proof: a proof that involves calculations and makes reference to the coordinate plane

equiangular: having equal angles
### Instruction

<table>
<thead>
<tr>
<th>Term</th>
<th>Definition</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>equilateral triangle</strong></td>
<td>a triangle with all three sides equal in length</td>
</tr>
<tr>
<td><strong>exterior angle of a polygon</strong></td>
<td>an angle formed by one side of a polygon and the extension of another side</td>
</tr>
<tr>
<td><strong>incenter</strong></td>
<td>the intersection of the angle bisectors of a triangle</td>
</tr>
<tr>
<td><strong>inscribed circle</strong></td>
<td>a circle that contains one point from each side of a triangle</td>
</tr>
<tr>
<td><strong>interior angle of a polygon</strong></td>
<td>an angle formed by two sides of a polygon</td>
</tr>
<tr>
<td><strong>isosceles triangle</strong></td>
<td>a triangle with at least two congruent sides</td>
</tr>
<tr>
<td><strong>legs</strong></td>
<td>congruent sides of an isosceles triangle</td>
</tr>
<tr>
<td><strong>median of a triangle</strong></td>
<td>the segment joining the vertex to the midpoint of the opposite side</td>
</tr>
<tr>
<td><strong>midpoint</strong></td>
<td>a point on a line segment that divides the segment into two equal parts</td>
</tr>
</tbody>
</table>
| **midpoint formula**    | a formula that states the midpoint of a segment created by connecting \((x_1, y_1)\) and \((x_2, y_2)\) is given by the formula \[
\left( \frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2} \right) \]
| **midsegment**          | a line segment joining the midpoints of two sides of a figure             |
| **midsegment triangle** | the triangle formed when all three of the midsegments of a triangle are connected |
| **obtuse triangle**     | a triangle with one angle that is obtuse (greater than 90°)              |
| **orthocenter**         | the intersection of the altitudes of a triangle                           |
| **point of concurrency**| a single point of intersection of three or more lines                    |
| **remote interior angles** | interior angles that are not adjacent to the exterior angle            |
| **right triangle**      | a triangle with one angle that measures 90°                              |
| **scalene triangle**    | a triangle with no congruent sides                                       |
| **supplementary angles**| two angles whose sum is 180°                                             |
| **vertex angle**        | angle formed by the legs of an isosceles triangle                         |
Recommended Resources

  http://www.walch.com/rr/00032
  This interactive website provides a series of problems related to midsegments of triangles and scores them immediately. If the user submits a wrong answer, a description and process for arriving at the correct answer are given.

  http://www.walch.com/rr/00033
  This website gives a brief explanation of the centroid of a triangle. An interactive applet allows users to change the size and shape of a triangle and observe the changes in the centroid. Also included are links to summaries of each of the triangle centers.

  http://www.walch.com/rr/00034
  This website gives a brief explanation of the properties of isosceles triangles. Also included are links to finding the centers of triangles, as well as an interactive illustration demonstrating isosceles triangle properties.

  http://www.walch.com/rr/00035
  This website gives a brief explanation of the properties of triangles, including interior and exterior angles. The site also contains an interactive applet that allows users to change the measure of one angle of a triangle and observe the changes in the remaining angles.

  http://www.walch.com/rr/00036
  This website contains a chart of the centers of a triangles and the lines used to find each center, as well as where the center is located on various triangles. Also included are links to interactive applets for each center.
Lesson 1.9.1: Proving the Interior Angle Sum Theorem

Warm-Up 1.9.1

When a beam of light is reflected from a flat surface, the angle of incidence is congruent to the angle of reflection. The diagram below shows a ray of light from a flashlight being reflected off a mirror. Use the diagram to answer the questions that follow.

1. What is the measure of the angle of reflection? Explain how you found your answer.

2. What is the measure of the angle created by the mirror and the flashlight? Explain how you found your answer.

3. Describe how to determine the measure of the angle created by the mirror and the reflected ray of light.
Lesson 1.9.1: Proving the Interior Angle Sum Theorem

Common Core Georgia Performance Standard
MCC9–12.G.CO.10

Warm-Up 1.9.1 Debrief

When a beam of light is reflected from a flat surface, the angle of incidence is congruent to the angle of reflection. The diagram below shows a ray of light from a flashlight being reflected off a mirror. Use the diagram to answer the questions that follow.

1. What is the measure of the angle of reflection? Explain how you found your answer.
   The measure of the angle of reflection is congruent to the angle of incidence.
   The angle of incidence is 70°, so the angle of reflection is also 70°.

2. What is the measure of the angle created by the mirror and the flashlight? Explain how you found your answer.
   The angle created by the reflected ray and the angle of reflection is given as a right angle.
   The angle created by the mirror and the flashlight and the angle of incidence is complementary to that of the angle of reflection and the reflected ray; that is, it’s equal to 90°.
   The angle of incidence is 70°.
To find the measure of the angle created by the mirror and the flashlight, subtract 70° from 90°.

\[ 90 - 70 = 20 \]

The measure of the angle is 20°.

3. Describe how to determine the measure of the angle created by the mirror and the reflected ray of light.

There are several ways to determine the measure of the angle created by the mirror and the reflected ray.

The angle created by the mirror and the reflected ray is congruent to the angle created by the mirror and the incident ray.

This is true because we are told that the angle of incidence is congruent to the angle of reflection.

Since the angle of incidence is 70°, the angle of reflection is also 70°.

The angle created by the mirror and the flashlight is the complement of the angle of incidence.

The angle created by the mirror and the reflected ray is also the complement to the angle of reflection.

By subtracting the angle of reflection (70°) from 90°, we are able to determine that the measure of the angle created by the mirror and the reflected ray is 20°.

**Connection to the Lesson**

- Students will continue to find unknown angle measures using previously learned angle relationships.
- Students’ understanding of angle relationships will be expanded to include angles of triangles.
Introduction

Think of all the different kinds of triangles you can create. What are the similarities among the triangles? What are the differences? Are there properties that hold true for all triangles and others that only hold true for certain types of triangles? This lesson will explore angle relationships of triangles. We will examine the relationships of interior angles of triangles as well as the exterior angles of triangles, and how these relationships can be used to find unknown angle measures.

Key Concepts

• There is more to a triangle than just three sides and three angles.
• Triangles can be classified by their angle measures or by their side lengths.
• Triangles classified by their angle measures can be acute, obtuse, or right triangles.
• All of the angles of an **acute triangle** are acute, or less than 90°.
• One angle of an **obtuse triangle** is obtuse, or greater than 90°.
• A **right triangle** has one angle that measures 90°.
• Triangles classified by the number of congruent sides can be scalene, isosceles, or equilateral.

• A **scalene triangle** has no congruent sides.

• An **isosceles triangle** has at least two congruent sides.

• An equilateral triangle has three congruent sides.

<table>
<thead>
<tr>
<th>Scalene triangle</th>
<th>Isosceles triangle</th>
<th>Equilateral triangle</th>
</tr>
</thead>
<tbody>
<tr>
<td><img src="image1" alt="Scalene Triangle" /></td>
<td><img src="image2" alt="Isosceles Triangle" /></td>
<td><img src="image3" alt="Equilateral Triangle" /></td>
</tr>
<tr>
<td>No congruent sides</td>
<td>At least two congruent sides</td>
<td>Three congruent sides</td>
</tr>
</tbody>
</table>

• It is possible to create many different triangles, but the sum of the angle measures of every triangle is 180°. This is known as the Triangle Sum Theorem.

### Theorem

**Triangle Sum Theorem**

The sum of the angle measures of a triangle is 180°.

\[ m \angle A + m \angle B + m \angle C = 180 \]

• The Triangle Sum Theorem can be proven using the Parallel Postulate.

• The Parallel Postulate states that if a line can be created through a point not on a given line, then that line will be parallel to the given line.
• This postulate allows us to create a line parallel to one side of a triangle to prove angle relationships.

### Postulate

**Parallel Postulate**

Given a line and a point not on it, there exists one and only one straight line that passes through that point and never intersects the first line.

• This theorem can be used to determine a missing angle measure by subtracting the known measures from 180°.

• Most often, triangles are described by what is known as the **interior angles** of triangles (the angles formed by two sides of the triangle), but exterior angles also exist.

• In other words, interior angles are the angles inside the triangle.

• **Exterior angles** are angles formed by one side of the triangle and the extension of another side.

• The interior angles that are not adjacent to the exterior angle are called the **remote interior angles** of the exterior angle.
• The following illustration shows the differences among interior angles, exterior angles, and remote interior angles.

![Diagram of triangle with labeled angles](image)

• Interior angles: \( \angle A, \angle B, \) and \( \angle C \)
• Exterior angle: \( \angle D \)
• Remote interior angles of \( \angle D \): \( \angle A \) and \( \angle B \)
• Notice that \( \angle C \) and \( \angle D \) are supplementary; that is, together they create a line and sum to 180°.
• The measure of an exterior angle is equal to the sum of the measure of its remote interior angles. This is known as the Exterior Angle Theorem.

### Theorem

**Exterior Angle Theorem**

The measure of an exterior angle of a triangle is equal to the sum of the measures of its remote interior angles.

\[
m \angle D = m \angle A + m \angle B
\]
This theorem can also be used to determine a missing angle measure of a triangle.

The measure of an exterior angle will always be greater than either of the remote interior angles. This is known as the Exterior Angle Inequality Theorem.

The following theorems are also helpful when finding the measures of missing angles and side lengths.

**Theorem**

**Exterior Angle Inequality Theorem**

If an angle is an exterior angle of a triangle, then its measure is greater than the measure of either of its corresponding remote interior angles.

![Diagram of a triangle with exterior angle](image)

\[ m\angle D > m\angle A \]

\[ m\angle D > m\angle B \]

**Theorem**

If one side of a triangle is longer than another side, then the angle opposite the longer side has a greater measure than the angle opposite the shorter side.
Theorem

If one angle of a triangle has a greater measure than another angle, then the side opposite the greater angle is longer than the side opposite the lesser angle.

\[ m\angle A < m\angle B < m\angle C \]
\[ a < b < c \]

- The Triangle Sum Theorem and the Exterior Angle Theorem will be proven in this lesson.

Common Errors/Misconceptions
- incorrectly identifying the remote interior angles associated with an exterior angle
- incorrectly applying theorems to determine missing angle measures
- misidentifying or leaving out theorems, postulates, or definitions when writing proofs
Guided Practice 1.9.1

Example 1

Find the measure of $\angle C$.

1. Identify the known information.

Two measures of the three interior angles are given in the problem.

$m\angle A = 80$

$m\angle B = 65$

The measure of $\angle C$ is unknown.

2. Calculate the measure of $\angle C$.

The sum of the measures of the interior angles of a triangle is $180^\circ$.

Create an equation to solve for the unknown measure of $\angle C$.

$m\angle A + m\angle B + m\angle C = 180$  \hspace{1cm} \text{Triangle Sum Theorem}

$80 + 65 + m\angle C = 180$  \hspace{1cm} \text{Substitute values for } m\angle A \text{ and } m\angle B.

$145 + m\angle C = 180$  \hspace{1cm} \text{Simplify.}

$m\angle C = 35$  \hspace{1cm} \text{Solve for } m\angle C.

3. State the answer.

The measure of $\angle C$ is $35^\circ$.  \hspace{1cm} \checkmark
Example 2
Find the missing angle measures.

1. Identify the known information.

The figure contains two triangles, \( \triangle ABC \) and \( \triangle CDE \).

The measures of two of the three interior angles of \( \triangle ABC \) are given in the problem.

- \( m \angle A = 50 \)
- \( m \angle B = 55 \)

The measure of \( \angle BCA \) is unknown.

The measure of one of the three interior angles of \( \triangle CDE \) is given in the problem.

- \( m \angle E = 40 \)

The measures of \( \angle DCE \) and \( \angle D \) are unknown.
2. Calculate the unknown measures.

The sum of the measures of the interior angles of a triangle is 180°.

Create an equation to solve for the unknown measure of $\angle BCA$.

\[ m\angle A + m\angle B + m\angle BCA = 180 \]  
Triangle Sum Theorem

\[ 50 + 55 + m\angle BCA = 180 \]  
Substitute values for $m\angle A$ and $m\angle B$.

\[ 105 + m\angle BCA = 180 \]  
Simplify.

\[ m\angle BCA = 75 \]  
Solve for $m\angle BCA$.

$\angle BCA$ and $\angle DCE$ are vertical angles and are congruent.

\[ m\angle DCE = m\angle BCA = 75 \]

Create an equation to solve for the unknown measure of $\angle D$.

\[ m\angle DCE + m\angle D + m\angle E = 180 \]  
Triangle Sum Theorem

\[ 75 + m\angle D + 40 = 180 \]  
Substitute values for $m\angle DCE$ and $m\angle E$.

\[ 115 + m\angle D = 180 \]  
Simplify.

\[ m\angle D = 65 \]  
Solve for $m\angle D$.

3. State the answer.

The measure of $\angle BCA$ is 75°.

The measure of $\angle DCE$ is 75°.

The measure of $\angle D$ is 65°.
Example 3

Find the missing angle measures.

1. Identify the known information.

   One exterior angle measures 155°.
   The measures of the remote interior angles are stated as expressions.
   The value of $x$ is unknown.

2. Calculate the unknown measures.

   The measure of an exterior angle of a triangle is equal to the sum of the measures of its remote interior angles.

   Create an equation to solve for the unknown value of $x$.

   \[
   155 = (3x - 10) + (4x + 60) \quad \text{Exterior Angle Theorem}
   
   155 = 7x + 50 \quad \text{Simplify.}
   
   105 = 7x \quad \text{Solve for } x.
   
   x = 15
   \]
3. Determine the unknown measures using the value of $x$.

   $3x - 10$  
   $= 3(15) - 10$  
   $= 35$  

   $4x + 60$  
   $= 4(15) + 60$  
   $= 120$

Check that the measures are correct.

   $155 = (3x - 10) + (4x + 60)$  
   $155 = 35 + 120$  
   $155 = 155$

4. State the answer.

   The measures of the remote interior angles are 35° and 120°.
Example 4

The Triangle Sum Theorem states that the sum of the angle measures of a triangle is 180°. Write a two-column proof of this theorem.

1. State the given information.
   
   Given: \( \triangle ABC \)

2. Draw a line, \( \ell \), through point \( B \) that is parallel to \( AC \) to aid in proving this theorem.

   \( \ell \parallel AC \)
   
   \( AB \) and \( BC \) are both transversals (lines that intersect a set of lines).
3. State known information about angles created by parallel lines cut by a transversal.

Alternate interior angles are congruent.

\( \angle 1 \) and \( \angle 4 \) are alternate interior angles and are congruent.

\( \angle 3 \) and \( \angle 5 \) are alternate interior angles and are congruent.

4. Identify known information about congruent angles.

Congruent angles have the same measure.

\( m \angle 1 = m \angle 4 \)
\( m \angle 3 = m \angle 5 \)

5. Write the information in a two-column proof.

<table>
<thead>
<tr>
<th>Statements</th>
<th>Reasons</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. ( \triangle ABC )</td>
<td>1. Given</td>
</tr>
<tr>
<td>2. Draw line ( \ell ) through ( B ) parallel to ( AC )</td>
<td>2. Parallel Postulate</td>
</tr>
<tr>
<td>3. ( \angle 1 \cong \angle 4 )</td>
<td>3. Alternate Interior Angles Theorem</td>
</tr>
<tr>
<td>4. ( \angle 3 \cong \angle 5 )</td>
<td>4. Alternate Interior Angles Theorem</td>
</tr>
<tr>
<td>5. ( m \angle 1 = m \angle 4 )</td>
<td>5. Definition of congruent angles</td>
</tr>
<tr>
<td>6. ( m \angle 3 = m \angle 5 )</td>
<td>6. Definition of congruent angles</td>
</tr>
<tr>
<td>7. ( m \angle 4 + m \angle 2 + m \angle 5 = 180 )</td>
<td>7. Angle Addition Postulate and the definition of a straight angle</td>
</tr>
<tr>
<td>8. ( m \angle 1 + m \angle 2 + m \angle 3 = 180 )</td>
<td>8. Substitution</td>
</tr>
</tbody>
</table>
Problem-Based Task 1.9.1: Sensing Distance

Distance-measuring sensors frequently used in robotics send out a beam of infrared light that hits an object. The beam bounces off the object and returns to the sensor’s detector, creating a triangle similar to the one below.

The angles of the triangle vary depending on the sensor’s distance from the object. The sensor uses the angles to determine how far away the object is. As the angle of reflection increases, the calculated distance becomes more accurate. In each diagram below, the sensor is parallel to the object. Which of the sensors calculates a more accurate distance: Sensor A, or Sensor B? Explain your reasoning.
Problem-Based Task 1.9.1: Sensing Distance

Coaching

Use the diagram below to answer the questions that follow.

a. What is the relationship between $\angle a$ and $\angle d$?

b. If the measure of $\angle d$ is 77°, what is the measure of $\angle a$?

c. What is the sum of the interior angles of a triangle?

d. What is the measure of $\angle c$?

e. What is the relationship between $\angle z$ and $\angle w$?

f. If the measure of $\angle z$ is 84°, what is the measure of $\angle w$?

g. What is the measure of $\angle y$?

h. Which sensor has a greater angle of reflection?

i. Which sensor is more accurate?
Problem-Based Task 1.9.1: Sensing Distance
Coaching Sample Responses

a. What is the relationship between $\angle a$ and $\angle d$?
   The sensor and the object are parallel; therefore, the segment from the sensor to the object acts as a transversal.
   $\angle a$ and $\angle d$ are alternate interior angles and are congruent.

b. If the measure of $\angle d$ is 77°, what is the measure of $\angle a$?
   The angles are congruent; therefore, $\angle a$ is also equal to 77°.

c. What is the sum of the interior angles of a triangle?
   The sum of the interior angles of any triangle is 180°.

d. What is the measure of $\angle c$?
   To determine the measure of $\angle c$, subtract the sum of $\angle a$ and $\angle b$ from 180.
   $180 - (77 + 77) = 26$
   The measure of $\angle c$ is 26°.
e. What is the relationship between $\angle z$ and $\angle w$?

$\angle z$ and $\angle w$ are alternate interior angles and are congruent.

f. If the measure of $\angle z$ is $84^\circ$, what is the measure of $\angle w$?

The angles are congruent; therefore $\angle w$ is also equal to $84^\circ$.

g. What is the measure of $\angle y$?

To determine the measure of $\angle y$, subtract the sum of $\angle w$ and $\angle x$ from $180$.

Since the triangle is an isosceles triangle, $m\angle x = m\angle w = 84$.

$180 - (84 + 84) = 12$

The measure of $\angle y$ is $12^\circ$.

h. Which sensor has a greater angle of reflection?

The angle of reflection for Sensor A is $26^\circ$.

The angle of reflection for Sensor B is $12^\circ$.

Sensor A has a greater angle of reflection.

i. Which sensor is more accurate?

As the angle of reflection increases, the calculated distance is more accurate.

Sensor A has a greater angle of reflection; therefore, it is more accurate.

**Recommended Closure Activity**

Select one or more of the essential questions for a class discussion or as a journal entry prompt.
Use what you know about the sums of the interior and exterior angles of triangles to determine the measure of each identified angle.

1. Find $m\angle B$.

2. Find $m\angle C$.

3. Find $m\angle A$ and $m\angle B$. 

continued
4. Find \( m \angle A \), \( m \angle B \), and \( m \angle C \).

5. Find \( m \angle A \) and \( m \angle B \).

6. Find \( m \angle A \) and \( m \angle B \).
7. Find $m \angle CAB$ and $m \angle ABC$.

8. Find $m \angle CAB$ and $m \angle ABC$.

9. Find $m \angle CAB$ and $m \angle ABC$.

10. The Triangle Sum Theorem states that the sum of the angle measures of a triangle is $180^\circ$. Write a paragraph proof of this theorem, referring to the diagram below.
In the diagram below, a captain is aboard a ship at point $B$, a lighthouse is located at point $C$, and the ship is sailing in the direction of $\overrightarrow{BD}$. There is a buoy floating in the water at point $A$. The captain measured $\angle CBD$ to be $47^\circ$. The ship is as far from the buoy as the buoy is from the lighthouse, so the measures of $AB$ and $AC$ are equal.

1. What kind of triangle is $\triangle ABC$? Explain your reasoning.

2. What is the measure of $\angle ACB$?

3. What is the measure of $\angle CAD$?
Lesson 1.9.2: Proving Theorems About Isosceles Triangles

Common Core Georgia Performance Standard
MCC9–12.G.CO.10

Warm-Up 1.9.2 Debrief

In the diagram below, a captain is aboard a ship at point \( B \), a lighthouse is located at point \( C \), and the ship is sailing in the direction of \( \overrightarrow{BD} \). There is a buoy floating in the water at point \( A \). The captain measured \( \angle CBD \) to be 47°. The ship is as far from the buoy as the buoy is from the lighthouse, so the measures of \( AB \) and \( AC \) are equal.

1. What kind of triangle is \( \triangle ABC \)? Explain your reasoning.

\( \triangle ABC \) has two congruent sides, so by definition, \( \triangle ABC \) is an isosceles triangle.

2. What is the measure of \( \angle ACB \)?

By definition, an isosceles triangle has two congruent sides and two congruent angles. If \( AB \) is congruent to \( AC \), then the measure of \( \angle C \) is congruent to the measure of \( \angle B \).

\( \angle B \) measures 47°; therefore, \( \angle C \) also measures 47°.
3. What is the measure of $\angle CAD$?

$\angle C$ and $\angle B$ are remote interior angles to $\angle CAD$, an exterior angle.

The measure of an exterior angle is equal to the sum of the measure of its remote interior angles.

The sum of the interior angles is found by adding $m\angle C$ and $m\angle B$.

$47 + 47 = 94$

The measure of $\angle CAD$ is 94˚.

**Connection to the Lesson**

- Students will expand upon their understanding of isosceles triangles to find unknown side lengths and angle measures.
- Students will continue to use the Exterior Angle Theorem to find the unknown measures.
Introduction

Isosceles triangles can be seen throughout our daily lives in structures, supports, architectural details, and even bicycle frames. Isosceles triangles are a distinct classification of triangles with unique characteristics and parts that have specific names. In this lesson, we will explore the qualities of isosceles triangles.

Key Concepts

- Isosceles triangles have at least two congruent sides, called legs.
- The angle created by the intersection of the legs is called the vertex angle.
- Opposite the vertex angle is the base of the isosceles triangle.
- Each of the remaining angles is referred to as a base angle. The intersection of one leg and the base of the isosceles triangle creates a base angle.
The following theorem is true of every isosceles triangle.

**Theorem**

**Isosceles Triangle Theorem**

If two sides of a triangle are congruent, then the angles opposite the congruent sides are congruent.
• If the Isosceles Triangle Theorem is reversed, then that statement is also true.
• This is known as the converse of the Isosceles Triangle Theorem.

**Theorem**

**Converse of the Isosceles Triangle Theorem**
If two angles of a triangle are congruent, then the sides opposite those angles are congruent.

If the vertex angle of an isosceles triangle is bisected, the bisector is perpendicular to the base, creating two right triangles.
• In the diagram that follows, $D$ is the midpoint of $BC$.

• Equilateral triangles are a special type of isosceles triangle, for which each side of the triangle is congruent.

• If all sides of a triangle are congruent, then all angles have the same measure.

**Theorem**

If a triangle is equilateral then it is **equiangular**, or has equal angles.
Each angle of an equilateral triangle measures $60^\circ$ ($180^\circ \div 3 = 60^\circ$).
Conversely, if a triangle has equal angles, it is equilateral.

**Theorem**
If a triangle is equiangular, then it is equilateral.

\[ \overline{AB} \cong \overline{BC} \cong \overline{AC} \]

These theorems and properties can be used to solve many triangle problems.

**Common Errors/Misconceptions**
- incorrectly identifying parts of isosceles triangles
- not identifying equilateral triangles as having the same properties of isosceles triangles
- incorrectly setting up and solving equations to find unknown measures of triangles
- misidentifying or leaving out theorems, postulates, or definitions when writing proofs
Example 1

Find the measure of each angle of $\triangle ABC$.

1. Identify the congruent angles.
   
   The legs of an isosceles triangle are congruent; therefore, $\overline{AB} \cong \overline{AC}$.
   
   The base of $\triangle ABC$ is $\overline{BC}$.
   
   $\angle B$ and $\angle C$ are base angles and are congruent.

2. Calculate the value of $x$.

   Congruent angles have the same measure.

   Create an equation.
   
   $m\angle B = m\angle C$  
   
   The measures of base angles of isosceles triangles are equal.
   
   $4x = 6x - 36$  
   
   Substitute values for $m\angle B$ and $m\angle C$.
   
   $-2x = -36$  
   
   Solve for $x$.
   
   $x = 18$
3. Calculate each angle measure.

\[ m\angle B = 4x = 4(18) = 72 \]  
Substitute the value of \( x \) into the expression for \( m\angle B \).

\[ m\angle C = 6(18) - 36 = 72 \]  
Substitute the value of \( x \) into the expression for \( m\angle C \).

\[ m\angle A + m\angle B + m\angle C = 180 \]  
The sum of the angles of a triangle is 180°.

\[ m\angle A + 72 + 72 = 180 \]  
Substitute the known values.

\[ m\angle A = 36 \]  
Solve for \( m\angle A \).

4. Summarize your findings.

\[ m\angle A = 36 \]

\[ m\angle B = 72 \]

\[ m\angle C = 72 \]
Example 2

Determine whether \( \triangle ABC \) with vertices \( A (–4, 5) \), \( B (–1, –4) \), and \( C (5, 2) \) is an isosceles triangle. If it is isosceles, name a pair of congruent angles.

1. Use the distance formula to calculate the length of each side.

Calculate the length of \( AB \).

\[
d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}
\]

\[
AB = \sqrt{((-1) - (-4))^2 + ((-4) - (5))^2}
\]

Substitute \((-4, 5)\) and \((-1, -4)\) for \((x_1, y_1)\) and \((x_2, y_2)\).

\[
AB = \sqrt{3^2 + (-9)^2}
\]

Simplify.

\[
AB = \sqrt{9 + 81} = \sqrt{90} = 3\sqrt{10}
\]

Calculate the length of \( BC \).

\[
d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}
\]

\[
BC = \sqrt{((5) - (–1))^2 + (2) - (–4))^2}
\]

Substitute \((-1, -4)\) and \((5, 2)\) for \((x_1, y_1)\) and \((x_2, y_2)\).

\[
BC = \sqrt{(6)^2 + (6)^2}
\]

Simplify.

\[
BC = \sqrt{36 + 36} = \sqrt{72} = 6\sqrt{2}
\]

Calculate the length of \( AC \).

\[
d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}
\]

\[
AC = \sqrt{((5) - (–4))^2 + (2) - (–5))^2}
\]

Substitute \((-4, 5)\) and \((5, 2)\) for \((x_1, y_1)\) and \((x_2, y_2)\).

\[
AC = \sqrt{(9)^2 + (3)^2}
\]

Simplify.

\[
AC = \sqrt{81 + 9} = \sqrt{90} = 3\sqrt{10}
\]
2. Determine if the triangle is isosceles.
   A triangle with at least two congruent sides is an isosceles triangle. 
   \[ AB \cong AC \], so \( \triangle ABC \) is isosceles.

3. Identify congruent angles.
   If two sides of a triangle are congruent, then the angles opposite the sides are congruent. 
   \( \angle B \cong \angle C \)

Example 3
Given \( AB \cong AC \), prove that \( \angle B \cong \angle C \).

1. State the given information.
   \[ AB \cong AC \]
2. Draw the angle bisector of $\angle A$ and extend it to $BC$, creating the perpendicular bisector of $BC$. Label the point of intersection $D$.

Indicate congruent sides.

$\angle B$ and $\angle C$ are congruent corresponding parts.

3. Write the information in a two-column proof.

<table>
<thead>
<tr>
<th>Statements</th>
<th>Reasons</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. $\overline{AB} \cong \overline{AC}$</td>
<td>1. Given</td>
</tr>
<tr>
<td>2. Draw the angle bisector of $\angle A$ and extend it to $BC$, creating a perpendicular bisector of $BC$ and the midpoint of $BC$.</td>
<td>2. There is exactly one line through two points.</td>
</tr>
<tr>
<td>3. $\overline{BD} \cong \overline{BC}$</td>
<td>3. Definition of midpoint</td>
</tr>
<tr>
<td>4. $\overline{AD} \cong \overline{AD}$</td>
<td>4. Reflexive Property</td>
</tr>
<tr>
<td>5. $\triangle ABD \cong \triangle ACD$</td>
<td>5. SSS Congruence Statement</td>
</tr>
<tr>
<td>6. $\angle B \cong \angle C$</td>
<td>6. Corresponding Parts of Congruent Triangles are Congruent</td>
</tr>
</tbody>
</table>
Example 4

Find the values of $x$ and $y$.

1. Make observations about the figure.

   The triangle in the diagram has three congruent sides.
   A triangle with three congruent sides is equilateral.
   Equilateral triangles are also equiangular.
   The measure of each angle of an equilateral triangle is $60^\circ$.
   An exterior angle is also included in the diagram.
   The measure of an exterior angle is the supplement of the adjacent interior angle.
2. Determine the value of \( x \).
   
   The measure of each angle of an equilateral triangle is 60°.
   
   Create and solve an equation for \( x \) using this information.
   
   \[
   4x + 24 = 60 \quad \text{Equation}
   
   4x = 36 \quad \text{Solve for} \ x.
   
   x = 9
   
   The value of \( x \) is 9.
   
3. Determine the value of \( y \).
   
   The exterior angle is the supplement to the interior angle.
   
   The interior angle is 60° by the properties of equilateral triangles.
   
   The sum of the measures of an exterior angle and interior angle pair equals 180.
   
   Create and solve an equation for \( y \) using this information.
   
   \[
   11y - 23 + 60 = 180 \quad \text{Equation}
   
   11y + 37 = 180 \quad \text{Simplify.}
   
   11y = 143 \quad \text{Solve for} \ y.
   
   y = 13
   
   The value of \( y \) is 13.
Example 5

\( \triangle ABC \) is equilateral. Prove that it is equiangular.

1. State the given information.
   \( \triangle ABC \) is an equilateral triangle.

2. Plan the proof.
   Equilateral triangles are also isosceles triangles.
   Isosceles triangles have at least two congruent sides.
   \( AB \cong BC \)
   \( \angle A \) and \( \angle C \) are base angles in relation to \( AB \) and \( BC \).
   \( \angle A \cong \angle C \) because of the Isosceles Triangle Theorem.
   \( BC \cong AC \)
   \( \angle A \) and \( \angle B \) are base angles in relation to \( BC \) and \( AC \).
   \( \angle A \cong \angle B \) because of the Isosceles Triangle Theorem.
   By the Transitive Property, \( \angle A \cong \angle B \cong \angle C \); therefore, \( \triangle ABC \) is equiangular.

3. Write the information in a paragraph proof.
   Since \( \triangle ABC \) is equilateral, \( AB \cong BC \) and \( BC \cong AC \). By the Isosceles Triangle Theorem, \( \angle A \cong \angle C \) and \( \angle A \cong \angle B \). By the Transitive Property, \( \angle A \cong \angle B \cong \angle C \); therefore, \( \triangle ABC \) is equiangular.
Problem-Based Task 1.9.2: Calibrating Consoles

In order for a game console controller to work correctly, users must first calibrate the system. The user stands with the controller in front of the system and allows the system to recognize the controller’s location. Optimal performance is achieved when an isosceles triangle, as shown below, is created.

Scarlett and Wren have calibrated their game console controllers as shown in the following diagram. Did either of the girls calibrate her game console controller correctly? Explain your reasoning.
Problem-Based Task 1.9.2: Calibrating Consoles

Coaching

a. What is the definition of an isosceles triangle?

b. How can you determine the length of each side of the triangle Scarlett created?

c. What is the length of each side of the triangle Scarlett created?

d. Is the triangle Scarlett created an isosceles triangle?

e. How can you determine the measure of the third angle of the triangle Wren created?

f. What is the measure of the third angle of the triangle Wren created?

g. Is the triangle Wren created an isosceles triangle?

h. Did either girl calibrate her game console controller correctly? If so, who?
**Problem-Based Task 1.9.2: Calibrating Consoles**

**Coaching Sample Responses**

a. What is the definition of an isosceles triangle?
   
   Isosceles triangles have at least two sides that are congruent and at least two base angles that are congruent.

b. How can you determine the length of each side of the triangle Scarlett created?
   
   The triangle Scarlett created is made up of two smaller triangles.
   
   The smaller triangles are congruent because of the Side-Angle-Side Congruence Statement.
   
   Use the Pythagorean Theorem to calculate the lengths of the sides of the triangle Scarlett created.

   c. What is the length of each side of the triangle Scarlett created?

   \[ a^2 + b^2 = c^2 \]
   
   Pythagorean Theorem

   \[ 5^2 + 12^2 = c^2 \]
   
   Substitute 5 and 12 for the lengths of \( a \) and \( b \).

   \[ 25 + 144 = c^2 \]
   
   Simplify.

   \[ 169 = c^2 \]

   \[ c = \sqrt{169} = 13 \]

   Remember that the two smaller triangles are congruent.

   The sides of the triangle Scarlett created are each 13 feet long.

d. Is the triangle Scarlett created an isosceles triangle?
   
   The triangle Scarlett created is an isosceles triangle because two of the sides are equal in length.

e. How can you determine the measure of the third angle of the triangle Wren created?
   
   The sum of the interior angles of a triangle is 180°.
   
   Subtract the given angles from 180 to determine the unknown angle measure.
f. What is the measure of the third angle of the triangle Wren created?

   \[ 180 - (68 + 44) = 68 \]

   The third angle measure of the triangle Wren created is 68˚.

g. Is the triangle Wren created an isosceles triangle?

   The triangle Wren created is an isosceles triangle because two of the angles are congruent.

h. Did either girl calibrate her game console controller correctly? If so, who?

   Both girls calibrated their game console controllers correctly because the triangle created by each girl is an isosceles triangle.

**Recommended Closure Activity**

Select one or more of the essential questions for a class discussion or as a journal entry prompt.
Practice 1.9.2: Proving Theorems About Isosceles Triangles

Use what you know about isosceles triangles to find each angle measure.

1. \( m \angle B \) and \( m \angle C \)

\[
\begin{align*}
A & \quad 50^\circ \\
B & \\
C & \\
\end{align*}
\]

2. \( m \angle B \), \( m \angle C \), and \( m \angle D \)

\[
\begin{align*}
A & \quad 65^\circ \\
B & \\
C & \\
D & \\
\end{align*}
\]

3. \( m \angle A \), \( m \angle B \), and \( m \angle C \)

\[
\begin{align*}
A & \quad (4x + 6)^\circ \\
B & \quad (20x)^\circ \\
C & \\
\end{align*}
\]
4. $m\angle A$, $m\angle B$, and $m\angle C$

Find each value using the given information.

5. $x$
6. \( m\angle x \) and \( m\angle y \)

7. \( x \) and \( y \)

For problems 8 and 9, use the given vertices to determine whether \( \triangle ABC \) is an isosceles triangle. If it is isosceles, name a pair of congruent angles.

8. \( A (0, 0), B (-8, 0), C (-4, -6) \)

9. \( A (1, 1), B (-4, 4), C (6, -2) \)
10. The converse of the Isosceles Triangle Theorem states that if two angles of a triangle are congruent, then the sides opposite those angles are congruent. Write a two-column proof of this statement.
Lesson 1.9.3: Proving the Midsegment of a Triangle

Warm-Up 1.9.3

A portion of the Los Angeles marathon course is mapped on the coordinate plane shown. Each unit represents 600 feet. The locations of major landmarks along the course are labeled by points.

1. Officials need to verify the distance between points C and D. What is this distance?

2. A water station is planned midway between points E and F. What is the location of the water station?
Lesson 1.9.3: Proving the Midsegment of a Triangle

Common Core Georgia Performance Standard
MCC9–12.G.CO.10

Warm-Up 1.9.3 Debrief

A portion of the Los Angeles marathon course is mapped on the coordinate plane shown. Each unit represents 600 feet. The locations of major landmarks along the course are labeled by points.

1. Officials need to verify the distance between points $C$ and $D$. What is this distance?

Point $C$ is located at $(-8, 2)$ and point $D$ is located at $(-6, -4)$.

To calculate the distance between points $C$ and $D$, use the distance formula.
Lesson 9: Proving Theorems About Triangles

**Instruction**

The distance formula is given by:

\[ d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \]

Substitute \((-8, 2)\) and \((-6, -4)\) for \((x_1, y_1)\) and \((x_2, y_2)\).

\[ d = \sqrt{[(-6) - (-8)]^2 + [(-4) - (2)]^2} \]

Simplify:

\[ d = \sqrt{4 + 36} \]

\[ d = \sqrt{40} \approx 6.3 \text{ units} \]

Each unit on the map is 600 feet.

To find the actual distance between points \(C\) and \(D\), multiply the number of units by 600.

\[ 6.3 \times 600 = 3780 \]

The distance between points \(C\) and \(D\) is approximately 3,780 feet.

2. A water station is planned midway between points \(E\) and \(F\). What is the location of the water station?

Point \(E\) is located at \((-2, -4)\) and point \(F\) is located at \((6, 6)\).

To calculate the point that is midway between points \(E\) and \(F\), use the midpoint formula.

\[ \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right) \]

Substitute \((-2, -4)\) and \((6, 6)\) for \((x_1, y_1)\) and \((x_2, y_2)\).

\[ \left(\frac{-2 + 6}{2}, \frac{-4 + 6}{2}\right) \]

Simplify:

\[ (2, 1) \]

The midpoint of \(EF\) is \((2, 1)\).

The water station is planned at the point \((2, 1)\).

**Connection to the Lesson**

- Students will expand upon their understanding of distance and midpoint of segments and apply this knowledge to triangles.
Prerequisite Skills
This lesson requires the use of the following skills:
- calculating the midpoint of a segment
- calculating slopes of lines
- determining if lines are parallel based on slopes
- writing various forms of proofs

Introduction
Triangles are typically thought of as simplistic shapes constructed of three angles and three segments. As we continue to explore this shape, we discover there are many more properties and qualities than we may have first imagined. Each property and quality, such as the midsegment of a triangle, acts as a tool for solving problems.

Key Concepts
- The midpoint is the point on a line segment that divides the segment into two equal parts.
- A midsegment of a triangle is a line segment that joins the midpoints of two sides of a triangle.

- In the diagram above, the midpoint of $\overline{AB}$ is $X$.
- The midpoint of $\overline{BC}$ is $Y$.
- A midsegment of $\triangle ABC$ is $\overline{XY}$. 
• The midsegment of a triangle is parallel to the third side of the triangle and is half as long as the third side. This is known as the Triangle Midsegment Theorem.

**Theorem**

**Triangle Midsegment Theorem**

A midsegment of a triangle is parallel to the third side and is half as long.

\[
\overline{AC} \parallel \overline{XY}
\]

\[
XY = \frac{1}{2} AC
\]

• Every triangle has three midsegments.
When all three of the midsegments of a triangle are connected, a **midsegment triangle** is created.

In the diagram above, \( \triangle ABC \sim \triangle TSR \).

**Coordinate proofs**, proofs that involve calculations and make reference to the coordinate plane, are often used to prove many theorems.

### Common Errors/Misconceptions
- assuming a segment that is parallel to the third side of a triangle is a midsegment
- incorrectly writing and solving equations to determine lengths
- incorrectly calculating slope
- incorrectly applying the Triangle Midsegment Theorem to solve problems
- misidentifying or leaving out theorems, postulates, or definitions when writing proofs
Example 1

Find the lengths of $BC$ and $YZ$ and the measure of $\angle AXZ$.

1. **Identify the known information.**
   - Tick marks indicate that $X$ is the midpoint of $\overline{AB}$, $Y$ is the midpoint of $\overline{BC}$, and $Z$ is the midpoint of $\overline{AC}$.
   - $\overline{XZ}$ and $\overline{YZ}$ are midsegments of $\triangle ABC$.

2. **Calculate the length of $BC$.**
   - $\overline{XZ}$ is the midsegment that is parallel to $\overline{BC}$.
   - The length of $\overline{XZ}$ is $\frac{1}{2}$ the length of $\overline{BC}$.
   - $XZ = \frac{1}{2} BC$  
   - $4.8 = \frac{1}{2} BC$  
   - $4.8 = \frac{1}{2} BC$  
   - Substitute 4.8 for $XZ$.
   - $BC = 9.6$  
   - Solve for $BC$. 
3. Calculate the measure of $YZ$.

   $YZ$ is the midsegment parallel to $AB$.
   The length of $YZ$ is $\frac{1}{2}$ the length of $AB$.

   \[
   YZ = \frac{1}{2} AB \quad \text{Triangle Midsegment Theorem}
   \]

   \[
   YZ = \frac{1}{2} (11.5) \quad \text{Substitute 11.5 for } AB.
   \]

   \[
   YZ = 5.75 \quad \text{Solve for } YZ.
   \]

4. Calculate the measure of $\angle AXZ$.

   $YZ \parallel AB$ \hspace{1cm} \text{Triangle Midsegment Theorem}

   \[
   m\angle AXZ = m\angle XZY \quad \text{Alternate Interior Angles Theorem}
   \]

   $m\angle AXZ = 38$

5. State the answers.

   $BC$ is 9.6 units long. $YZ$ is 5.75 units long. $m\angle AXZ$ is 38°.
Example 2

If $AB = 2x + 7$ and $YZ = 3x - 6.5$, what is the length of $AB$?

1. Identify the known information.

   Tick marks indicate that $X$ is the midpoint of $AB$, $Y$ is the midpoint of $BC$, and $Z$ is the midpoint of $AC$.

   $XY$, $XZ$, and $YZ$ are the midsegments of $\triangle ABC$. 

2. Calculate the length of $AB$.
   The length of $YZ$ is $\frac{1}{2}$ the length of $AB$.
   
   $YZ = \frac{1}{2} AB$  
   Triangle Midsegment Theorem

   $3x - 6.5 = \frac{1}{2}(2x + 7)$  
   Substitute values for $YZ$ and $AB$.

   $3x - 6.5 = x + 3.5$  
   Solve for $x$.

   $2x - 6.5 = 3.5$  
   $2x = 10$
   $x = 5$

   Use the value of $x$ to find the length of $AB$.
   
   $AB = 2x + 7$
   $= 2(5) + 7$
   $= 10 + 7$
   $= 17$

3. State the answer.
   
   $AB = 17$
Example 3

The midpoints of a triangle are $X(-2, 5)$, $Y(3, 1)$, and $Z(4, 8)$. Find the coordinates of the vertices of the triangle.

1. Plot the midpoints on a coordinate plane.
2. Connect the midpoints to form the midsegments $XY$, $YZ$, and $XZ$. 
3. Calculate the slope of each midsegment.

   Calculate the slope of \( \overline{XY} \).
   
   \[
   m = \frac{y_2 - y_1}{x_2 - x_1}
   \]
   
   Slope formula
   
   \[
   m = \frac{(1) - (5)}{(3) - (-2)}
   \]
   
   Substitute \((-2, 5)\) and \((3, 1)\) for \((x_1, y_1)\) and \((x_2, y_2)\).
   
   \[
   m = -\frac{4}{5}
   \]
   
   Simplify.
   
   The slope of \( \overline{XY} \) is \(-\frac{4}{5}\).

   Calculate the slope of \( \overline{YZ} \).
   
   \[
   m = \frac{y_2 - y_1}{x_2 - x_1}
   \]
   
   Slope formula
   
   \[
   m = \frac{(8) - (1)}{(4) - (3)}
   \]
   
   Substitute \((3, 1)\) and \((4, 8)\) for \((x_1, y_1)\) and \((x_2, y_2)\).
   
   \[
   m = -\frac{7}{1}
   \]
   
   Simplify.
   
   The slope of \( \overline{YZ} \) is 7.

   Calculate the slope of \( \overline{XZ} \).
   
   \[
   m = \frac{y_2 - y_1}{x_2 - x_1}
   \]
   
   Slope formula
   
   \[
   m = \frac{(8) - (5)}{(4) - (-2)}
   \]
   
   Substitute \((-2, 5)\) and \((4, 8)\) for \((x_1, y_1)\) and \((x_2, y_2)\).
   
   \[
   m = -\frac{3}{6}
   \]
   
   Simplify.
   
   The slope of \( \overline{XZ} \) is \(-\frac{1}{2}\).
4. Draw the lines that contain the midpoints.

The endpoints of each midsegment are the midpoints of the larger triangle.

Each midsegment is also parallel to the opposite side.

The slope of \( \overline{XZ} \) is \( \frac{1}{2} \).

From point \( Y \), draw a line that has a slope of \( \frac{1}{2} \).
The slope of $YZ$ is 7.

From point $X$, draw a line that has a slope of 7.
The slope of $\overline{XY}$ is $-\frac{4}{5}$. From point $Z$, draw a line that has a slope of $-\frac{4}{5}$.

The intersections of the lines form the vertices of the triangle.
5. Determine the vertices of the triangle.

The vertices of the triangle are (–3, –2), (9, 4), and (–1, 12).
Example 4

Write a coordinate proof of the Triangle Midsegment Theorem using the graph below.

1. Identify the known information.
   According to the graph, \( \triangle ABC \) has vertices located at \((2a, 2b)\), \((0, 0)\), and \((2c, 0)\).
2. Find the coordinates of \( D \), which is the midpoint of \( \overline{AB} \).

Use the midpoint formula to find the midpoint of \( \overline{AB} \).

\[
\left( \frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2} \right)
\]
Midpoint formula

\[
\left( \frac{2a + 0}{2}, \frac{2b + 0}{2} \right)
\]
Substitute \((2a, 2b)\) and \((0, 0)\) for \((x_1, y_1)\) and \((x_2, y_2)\).

\[
\left( \frac{2a}{2}, \frac{2b}{2} \right)
\]
Simplify.

\((a, b)\)

The midpoint of \( \overline{AB} \) is \((a, b)\).

3. Find the coordinates of \( E \), which is the midpoint of \( \overline{AC} \).

Use the midpoint formula to find the midpoint of \( \overline{AC} \).

\[
\left( \frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2} \right)
\]
Midpoint formula

\[
\left( \frac{2a + 2c}{2}, \frac{2b + 0}{2} \right)
\]
Substitute \((2a, 2b)\) and \((2c, 0)\) for \((x_1, y_1)\) and \((x_2, y_2)\).

\[
\left( \frac{2a + 2c}{2}, \frac{2b}{2} \right)
\]
Simplify.

\((a + c, b)\)

The midpoint of \( \overline{AC} \) is \((a + c, b)\).
4. Calculate the slope of the midsegment.
   Use the slope formula to calculate the slope of $\overline{DE}$.
   \[
m = \frac{y_2 - y_1}{x_2 - x_1}
   \]
   Slope formula
   \[
m = \frac{(b) - (b)}{(a + c) - (a)}
   \]
   Substitute $(a, b)$ and $(a + c, b)$ for $(x_1, y_1)$ and $(x_2, y_2)$.
   \[
m = \frac{0}{c} = 0
   \]
   Simplify.
   The slope of $\overline{DE}$ is 0.

5. Calculate the slope of $\overline{BC}$.
   Use the slope formula to calculate the slope of $\overline{BC}$.
   \[
m = \frac{y_2 - y_1}{x_2 - x_1}
   \]
   Slope formula
   \[
m = \frac{(0) - (0)}{(2c) - (0)}
   \]
   Substitute $(0, 0)$ and $(2c, 0)$ for $(x_1, y_1)$ and $(x_2, y_2)$.
   \[
m = \frac{0}{2c} = 0
   \]
   Simplify.
   The slope of $\overline{BC}$ is 0.
6. Calculate the length of \( \overline{DE} \).

Use the distance formula to calculate the length of the segment.

\[
d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}
\]

Distance formula

\[
d = \sqrt{(a + c) - a]^2 + (b - b)^2}
\]

Substitute \((a, b)\) and \((a + c, b)\)

\[
d = \sqrt{c^2 + 0}
\]

Simplify.

\[
d = \sqrt{c^2}
\]

\[
d = c
\]

The distance of \( \overline{DE} \) is \( c \) units.

7. Calculate the length of \( \overline{BC} \).

Use the distance formula to calculate the length of the segment.

\[
d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}
\]

Distance formula

\[
d = \sqrt{(2c - 0)^2 + (0 - 0)^2}
\]

Substitute \((0, 0)\) and \((2c, 0)\)

\[
d = \sqrt{(2c)^2 + 0}
\]

Simplify.

\[
d = \sqrt{4c^2}
\]

\[
d = 2c
\]

The distance of \( \overline{BC} \) is \( 2c \) units.

8. State your conclusion.

The slopes of \( \overline{DE} \) and \( \overline{BC} \) are both 0; therefore, the segments are parallel.

The length of \( \overline{DE} \) is \( c \) units and the length of \( \overline{BC} \) is \( 2c \) units; therefore, \( \overline{DE} \) is \( \frac{1}{2} \) the length of \( \overline{BC} \).
Problem-Based Task 1.9.3: Life-Size Support

Alex is assembling a life-size cardboard cutout of his favorite musician. The package contains three pieces of cardboard: the cutout, which measures 66 inches; the horizontal support, which measures 15.5 inches; and the angled support, which attaches to the back of the cutout at a point that is \( \frac{2}{3} \) the length of the cutout from the floor. The horizontal support acts as the midsegment connecting the cutout and the angled support. What is the length of the angled support rounded to the nearest inch?
Problem-Based Task 1.9.3: Life-Size Support

Coaching

a. If the cutout measures 66 inches, what is \( \frac{2}{3} \) the length of the cutout?

b. The horizontal support acts as a midsegment connecting the cutout to the angled support. How does the length of the midsegment compare to the side it is parallel to?

c. If the horizontal support is 15.5 inches, what is the distance from the bottom of the angled support to the base of the cutout?

d. How can you determine the length of the angled support?

e. What is the length of the angled support rounded to the nearest inch?
Problem-Based Task 1.9.3: Life-Size Support

Coaching Sample Responses

a. If the cutout measures 66 inches, what is \( \frac{2}{3} \) the length of the cutout?

To calculate \( \frac{2}{3} \) the length of the cutout, multiply \( \frac{2}{3} \) by 66.

\[
\frac{2}{3} \times 66 = \frac{2}{3} \times 44 = 44
\]

\( \frac{2}{3} \) the length of the cutout is 44 inches.

b. The horizontal support acts as a midsegment connecting the cutout to the angled support. How does the length of the midsegment compare to the side it is parallel to?

The midsegment connects the midpoints of two sides of the triangle, is parallel to the third side, and is half as long as the third side.

c. If the horizontal support is 15.5 inches, what is the distance from the bottom of the angled support to the base of the cutout?

The horizontal support is a midsegment of the triangle; therefore, the support is half as long as the distance from the angled support to the base of the cutout.

Multiply 15.5 by 2 to find this distance.

\[
15.5 \times 2 = 31
\]

The distance from the angled support to the base of the cutout is 31 inches.

d. How can you determine the length of the angled support?

The cutout and the angled support create a right triangle, for which the angled support is the hypotenuse.

To find the length of the hypotenuse, use the Pythagorean Theorem.
e. What is the length of the angled support rounded to the nearest inch?

\[ a^2 + b^2 = c^2 \]  
Pythagorean Theorem

\[ 31^2 + 44^2 = c^2 \]  
Substitute 31 and 44 for the lengths of \( a \) and \( b \).

\[ 961 + 1936 = c^2 \]  
Simplify.

\[ 2897 = c^2 \]

\[ c = \sqrt{2897} = 54 \text{ inches} \]

The length of the angled support rounded to the nearest inch is 54 inches.

**Recommended Closure Activity**

Select one or more of the essential questions for a class discussion or as a journal entry prompt.
Use your knowledge of midsegments to solve each problem.

1. Find the lengths of $BC$ and $XZ$ and the measure of $\angle BZX$.

2. Find the lengths of $AC$ and $YZ$ and the measure of $\angle XZY$. 

continued
3. If $AB = 7x - 13$ and $YZ = 2x + 4$, what is the length of $YZ$?

4. If $BC = 5x + 0.75$ and $XY = 3x - 0.25$, what is the length of $BC$?
5. The midpoints of a triangle are \(X (-5, -4), Y (-3, 2),\) and \(Z (5, 2)\). Find the coordinates of the vertices of the triangle.

6. The vertices of a triangle are \(A (-5, -4), B (1, 10),\) and \(C (9, 0)\). Find the coordinates of the midpoints of the triangle.

7. Use a coordinate proof to show that \(EF \parallel BC\) and \(\frac{EF}{BC} = \frac{1}{2}\).
8. Use a coordinate proof to show that \( EF \parallel AB \) and \( EF = \frac{1}{2} AB \).
9. Determine the midpoints of $\overline{AC}$ and $\overline{BC}$. Label the points $E$ and $F$. Show that $\overline{EF} \parallel \overline{AB}$ and $\frac{1}{2} = \overline{EF} = \overline{AB}$.
10. Determine the midpoints of $\overline{AC}$ and $\overline{BC}$. Label the points $E$ and $F$. Show that $\overline{EF} \parallel \overline{AB}$ and $EF = \frac{1}{2} AB$. 
Lesson 1.9.4: Proving Centers of Triangles

Warm-Up 1.9.4

Each unit on the map below represents 100 yards.

1. Karoline’s house is located at the point (2, 5). Her driveway is perpendicular to Oak Street, which is represented by the equation \( y = 2x - 3 \). What is the equation of the line that represents Karoline’s driveway?

2. What is the length of Karoline’s driveway from her house to Oak Street?
Lesson 1.9.4: Proving Centers of Triangles

Common Core Georgia Performance Standard
MCC9–12.G.CO.10

Warm-Up 1.9.4 Debrief
Each unit on the map below represents 100 yards.
1. Karoline’s house is located at the point (2, 5). Her driveway is perpendicular to Oak Street, which is represented by the equation \( y = 2x - 3 \). What is the equation of the line that represents Karoline’s driveway?

The equation of the line that represents Oak Street is \( y = 2x - 3 \). Therefore, the slope of the line that represents Oak Street is 2.

A line perpendicular to Oak Street has a slope that is the opposite reciprocal of 2.

The slope of the line that is perpendicular is \( -\frac{1}{2} \).

Find the \( y \)-intercept of the line perpendicular to Oak Street.

\[
y - y_1 = m(x - x_1)
\]

Point-slope form of a line

\[
y - 5 = -\frac{1}{2}(x - 2)
\]

Substitute (2, 5) for \((x_1, y_1)\) and \(-\frac{1}{2}\) for \(m\).

\[
y - 5 = -\frac{1}{2}x + 1
\]

Simplify.

\[
y = -\frac{1}{2}x + 6
\]

The equation of the line perpendicular to Oak Street is \( y = -\frac{1}{2}x + 6 \).

The equation of the line that represents Karoline’s driveway is \( y = -\frac{1}{2}x + 6 \).
2. What is the length of Karoline’s driveway from her house to Oak Street?

Determine the point at which Karoline’s driveway intersects with Oak Street.

Set the equation representing Oak Street equal to the equation representing Karoline’s driveway and solve for $x$.

$$2x - 3 = -\frac{1}{2}x + 6$$

$$\frac{5}{2}x - 3 = 6$$

Add $-\frac{1}{2}x$ to both sides of the equation.
Add 3 to both sides of the equation.

\[
\frac{5}{2}x = 9
\]

\[
x = \frac{3}{5} \times 3.6
\]

Divide both sides by \(\frac{5}{2}\).

Substitute the value of \(x\) into either of the equations.

\[
y = 2x - 3
\]

Equation of Oak Street

\[
y = 2(3.6) - 3
\]

Substitute 3.6 for \(x\).

\[
y = 7.2 - 3
\]

Simplify.

\[
y = 4.2
\]

The point at which Karoline’s driveway intersects with Oak Street is (3.6, 4.2).

Use the distance formula to calculate the distance from Karoline’s house to the point of intersection with Oak Street.

\[
d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}
\]

Distance formula

\[
d = \sqrt{(3.6 - 2)^2 + (4.2 - 5)^2}
\]

Substitute (2, 5) and (3.6, 4.2) for \((x_1, y_1)\) and \((x_2, y_2)\).

\[
d = \sqrt{(1.6)^2 + (-0.8)^2}
\]

Simplify.

\[
d = \sqrt{2.56 + 0.64}
\]

\[
d = \sqrt{3.2}
\]

The distance from Karoline’s house to the point of intersection with Oak Street is \(\sqrt{3.2}\) units.

Each unit on the map represents 100 yards.

To find the actual length, multiply \(\sqrt{3.2}\) by 100.

\[
\sqrt{3.2} \times 100 = 178.9
\]

The length of Karoline’s driveway is approximately 178.9 yards.

**Connection to the Lesson**

- Students will use their understanding of slopes of perpendicular lines and distances of segments to determine centers of triangles.
**Prerequisite Skills**

This lesson requires the use of the following skills:

- identifying and determining perpendicular bisectors and angle bisectors
- identifying and determining altitudes and medians of triangles
- calculating midpoints of segments
- writing various forms of proofs

**Introduction**

Think about all the properties of triangles we have learned so far and all the constructions we are able to perform. What properties exist when the perpendicular bisectors of triangles are constructed? Is there anything special about where the angle bisectors of a triangle intersect? We know triangles have three altitudes, but can determining each one serve any other purpose? How can the midpoints of each side of a triangle help find the center of gravity of a triangle? Each of these questions will be answered as we explore the centers of triangles.

**Key Concepts**

- Every triangle has four centers.
- Each center is determined by a different **point of concurrency**—the point at which three or more lines intersect.
- These centers are the circumcenter, the incenter, the orthocenter, and the centroid.

**Circumcenters**

- The perpendicular bisector is the line that is constructed through the midpoint of a segment. In the case of a triangle, the perpendicular bisectors are the midpoints of each of the sides.
- The three perpendicular bisectors of a triangle are **concurrent**, or intersect at one point.
- This point of concurrency is called the **circumcenter** of the triangle.
The circumcenter of a triangle is equidistant, or the same distance, from the vertices of the triangle. This is known as the Circumcenter Theorem.

**Theorem**

**Circumcenter Theorem**

The circumcenter of a triangle is equidistant from the vertices of a triangle.

![Diagram showing the circumcenter](image)

The circumcenter of this triangle is at X.

\[XA = XB = XC\]

- The circumcenter can be inside the triangle, outside the triangle, or even on the triangle depending on the type of triangle.
- The circumcenter is inside acute triangles, outside obtuse triangles, and on the midpoint of the hypotenuse of right triangles.
• Look at the placement of the circumcenter, point $X$, in the following examples.

<table>
<thead>
<tr>
<th>Acute triangle</th>
<th>Obtuse triangle</th>
<th>Right triangle</th>
</tr>
</thead>
<tbody>
<tr>
<td><img src="image1" alt="Acute triangle diagram" /></td>
<td><img src="image2" alt="Obtuse triangle diagram" /></td>
<td><img src="image3" alt="Right triangle diagram" /></td>
</tr>
<tr>
<td>$X$ is inside the triangle.</td>
<td>$X$ is outside the triangle.</td>
<td>$X$ is on the midpoint of the hypotenuse.</td>
</tr>
</tbody>
</table>

• The circumcenter of a triangle is also the center of the circle that connects each of the vertices of a triangle. This is known as the circle that circumscribes the triangle.
Incenters

- The angle bisectors of a triangle are rays that cut the measure of each vertex in half.
- The three angle bisectors of a triangle are also concurrent.
- This point of concurrency is called the incenter of the triangle.
- The incenter of a triangle is equidistant from the sides of the triangle. This is known as the Incenter Theorem.

**Theorem**

**Incenter Theorem**

The incenter of a triangle is equidistant from the sides of a triangle.

The incenter of this triangle is at $X$.

$XT = XU = XV$
• The incenter is always inside the triangle.

<table>
<thead>
<tr>
<th>Acute triangle</th>
<th>Obtuse triangle</th>
<th>Right triangle</th>
</tr>
</thead>
</table>

• The incenter of a triangle is the center of the circle that connects each of the sides of a triangle. This is known as the circle that **inscribes** the triangle.

**Orthocenters**

• The altitudes of a triangle are the perpendicular lines from each vertex of the triangle to its opposite side, also called the height of the triangle.

• The three altitudes of a triangle are also concurrent.

• This point of concurrency is called the **orthocenter** of the triangle.

• The orthocenter can be inside the triangle, outside the triangle, or even on the triangle depending on the type of triangle.

• The orthocenter is inside acute triangles, outside obtuse triangles, and at the vertex of the right angle of right triangles.
Look at the placement of the orthocenter, point $X$, in the following examples.

<table>
<thead>
<tr>
<th>Acute triangle</th>
<th>Obtuse triangle</th>
<th>Right triangle</th>
</tr>
</thead>
<tbody>
<tr>
<td>$X$ is inside the triangle.</td>
<td>$X$ is outside the triangle.</td>
<td>$X$ is at the vertex of the right angle.</td>
</tr>
</tbody>
</table>

**Centroids**

- The medians of a triangle are segments that join the vertices of the triangle to the midpoint of the opposite sides.

- Every triangle has three medians.

- The three medians of a triangle are also concurrent.

- This point of concurrency is called the **centroid** of the triangle.
• The centroid is always located inside the triangle $\frac{2}{3}$ the distance from each vertex to the midpoint of the opposite side. This is known as the Centroid Theorem.

**Theorem**

**Centroid Theorem**

The centroid of a triangle is $\frac{2}{3}$ the distance from each vertex to the midpoint of the opposite side.

The centroid of this triangle is at point $X$.

$$AX = \frac{2}{3} AU; \quad BX = \frac{2}{3} BV; \quad CX = \frac{2}{3} CT$$
• The centroid is always located inside the triangle.

<table>
<thead>
<tr>
<th>Acute triangle</th>
<th>Obtuse triangle</th>
<th>Right triangle</th>
</tr>
</thead>
<tbody>
<tr>
<td><img src="image1.png" alt="Acute Triangle Diagram" /></td>
<td><img src="image2.png" alt="Obtuse Triangle Diagram" /></td>
<td><img src="image3.png" alt="Right Triangle Diagram" /></td>
</tr>
</tbody>
</table>

• The centroid is also called the center of gravity of a triangle because the triangle will always balance at this point.

• Each point of concurrency discussed is considered a center of the triangle.

• Each center serves its own purpose in design, planning, and construction.

<table>
<thead>
<tr>
<th>Center of triangle</th>
<th>Intersection of…</th>
</tr>
</thead>
<tbody>
<tr>
<td>Circumcenter</td>
<td>Perpendicular bisectors</td>
</tr>
<tr>
<td>Incenter</td>
<td>Angle bisectors</td>
</tr>
<tr>
<td>Orthocenter</td>
<td>Altitudes</td>
</tr>
<tr>
<td>Centroid</td>
<td>Medians</td>
</tr>
</tbody>
</table>

**Common Errors/Misconceptions**

• not recognizing that the circumcenter and orthocenter are outside of obtuse triangles

• incorrectly assuming that the perpendicular bisector of the side of a triangle will pass through the opposite vertex

• interchanging circumcenter, incenter, orthocenter, and centroid

• confusing medians with midsegments

• misidentifying the height of the triangle
Guided Practice 1.9.4

Example 1

\( \triangle ABC \) has vertices \( A (3, 3), B (7, 3), \) and \( C (3, \ -3). \) Justify that \( (5, 0) \) is the circumcenter of \( \triangle ABC. \)

1. Identify the known information.

\( \triangle ABC \) has vertices \( A (3, 3), B (7, 3), \) and \( C (3, \ -3). \)

The circumcenter is \( X (5, 0). \)

The circumcenter of a triangle is equidistant from the vertices of the triangle.
2. Determine the distance from the circumcenter to each of the vertices.

Use the distance formula to calculate the distance from $X$ to $A$.

\[ d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \quad \text{Distance formula} \]

\[ d = \sqrt{[(3) - (5)]^2 + [(3) - (0)]^2} \]

\[ d = \sqrt{(-2)^2 + (3)^2} \quad \text{Substitute (5, 0) and (3, 3) for } (x_1', y_1') \text{ and } (x_2', y_2'). \]

\[ d = \sqrt{4 + 9} \quad \text{Simplify.} \]

\[ d = \sqrt{13} \]

The distance from $X$ to $A$ is $\sqrt{13}$ units.

Use the distance formula to calculate the distance from $X$ to $B$.

\[ d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \quad \text{Distance formula} \]

\[ d = \sqrt{[(7) - (5)]^2 + [(3) - (0)]^2} \]

\[ d = \sqrt{(2)^2 + (3)^2} \quad \text{Substitute (5, 0) and (7, 3) for } (x_1', y_1') \text{ and } (x_2', y_2'). \]

\[ d = \sqrt{4 + 9} \quad \text{Simplify.} \]

\[ d = \sqrt{13} \]

The distance from $X$ to $B$ is $\sqrt{13}$ units.

(continued)
Use the distance formula to calculate the distance from $X$ to $C$.

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

Distance formula

$$d = \sqrt{[(3) - (5)]^2 + [(-3) - (0)]^2}$$
Substitute (5, 0) and (3, –3) for $(x_1, y_1)$ and $(x_2, y_2)$.

$$d = \sqrt{(-2)^2 + (3)^2}$$
Simplify.

$$d = \sqrt{4 + 9}$$

$$d = \sqrt{13}$$

The distance from $X$ to $C$ is $\sqrt{13}$ units.

3. State your conclusion.

$X(5, 0)$ is the circumcenter of $\triangle ABC$ with vertices $A(3, 3)$, $B(7, 3)$, and $C(3, –3)$ because the distance from $X$ to each of the vertices is $\sqrt{13}$ units.
Example 2

\( \triangle ABC \) has vertices \( A(1, 6) \), \( B(7, 6) \), and \( C(3, 2) \). Find the equation of each altitude of \( \triangle ABC \) to verify that \((3, 4)\) is the orthocenter of \( \triangle ABC \).

1. Identify known information.

   \( \triangle ABC \) has vertices \( A(1, 6) \), \( B(7, 6) \), and \( C(3, 2) \).

   The orthocenter is \( X(3, 4) \).

   The orthocenter of a triangle is the intersection of the altitudes of the triangle.
2. Determine the altitudes of \( \triangle ABC \).

Find an equation of the line that is perpendicular to \( BC \) and passes through \( A \).

Use the slope formula to calculate the slope of \( BC \).

\[
m = \frac{y_2 - y_1}{x_2 - x_1}
\]

Slope formula

\[
m = \frac{(2) - (6)}{(3) - (7)}
\]

Substitute \((7, 6)\) and \((3, 2)\) for \((x_1, y_1)\) and \((x_2, y_2)\).

\[
m = \frac{-4}{-4} = 1
\]

Simplify.

The slope of \( BC \) is 1.

The slope of the line that is perpendicular to \( BC \) has the opposite reciprocal of the slope of \( BC \).

The slope of the perpendicular line is –1.

Find the \( y \)-intercept of the altitude from point \( A \).

\[
y - y_1 = m(x - x_1)
\]

Point-slope form of a line

\[
y - 6 = -1(x - 1)
\]

Substitute \((1, 6)\) for \((x_1, y_1)\) and \(-1\) for \( m \).

\[
y - 6 = -1x + 1
\]

Simplify.

\[
y = -x + 7
\]

The equation of the altitude from point \( A \) to \( BC \) is \( y = -x + 7 \).

Find an equation of the line that is perpendicular to \( AC \) and passes through \( B \).

Use the slope formula to calculate the slope of \( AC \).

\[
m = \frac{y_2 - y_1}{x_2 - x_1}
\]

Slope formula

\[
m = \frac{(2) - (6)}{(3) - (1)}
\]

Substitute \((1, 6)\) and \((3, 2)\) for \((x_1, y_1)\) and \((x_2, y_2)\).

\[
m = \frac{-4}{2} = -2
\]

Simplify.

The slope of \( AC \) is –2.

(continued)
The slope of the line that is perpendicular to $\overline{AC}$ has the opposite reciprocal of the slope of $\overline{AB}$.

The slope of the perpendicular line is $\frac{1}{2}$.

Find the $y$-intercept of the altitude from point $B$.

$$y - y_1 = m(x - x_1) \quad \text{Point-slope form of a line}$$

$$y - 6 = \frac{1}{2}(x - 7) \quad \text{Substitute (7, 6) for } (x_1, y_1) \text{ and } \frac{1}{2} \text{ for } m.$$ 

$$y - 6 = \frac{1}{2}x - \frac{7}{2} \quad \text{Simplify.}$$

$$y = -\frac{1}{2}x + \frac{5}{2}$$

The equation of the altitude from point $B$ to $\overline{AC}$ is $y = -\frac{1}{2}x + \frac{5}{2}$.

Find an equation of the line that is perpendicular to $\overline{AB}$ and passes through $C$.

Use the slope formula to calculate the slope of $\overline{AB}$.

$$m = \frac{y_2 - y_1}{x_2 - x_1} \quad \text{Slope formula}$$

$$m = \frac{(6) - (6)}{(7) - (1)} \quad \text{Substitute (1, 6) and (7, 6) for } (x_1, y_1) \text{ and } (x_2, y_2).$$

$$m = \frac{0}{6} = 0 \quad \text{Simplify.}$$

The slope of $\overline{AB}$ is 0.

$\overline{AB}$ is a horizontal line; therefore, the altitude is vertical.

The vertical line that passes through point $C$ is $x = 3$.

The equation of the altitude from point $C$ to $\overline{AB}$ is $x = 3$. 
3. Verify that $X(3, 4)$ is the intersection of the three altitudes.

For $(3, 4)$ to be the intersection of the three altitudes, the point must satisfy each of the equations: $y = -x + 7$, $y = -\frac{x}{2} + \frac{5}{2}$, and $x = 3$.

\[ y = -x + 7 \quad \text{Equation of the altitude from } A \text{ through } BC \]
\[ (4) = -(3) + 7 \quad \text{Substitute } X(3, 4) \text{ for } (x, y). \]
\[ 4 = 4 \quad \text{Simplify.} \]

$(3, 4)$ satisfies the equation of the altitude from $A$ through $BC$.

\[ y = -\frac{x}{2} + \frac{5}{2} \quad \text{Equation of the altitude from } B \text{ through } AC \]
\[ (4) = \frac{1}{2}(3) + \frac{5}{2} \quad \text{Substitute } X(3, 4) \text{ for } (x, y). \]
\[ 4 = \frac{3}{2} + \frac{5}{2} \quad \text{Simplify.} \]
\[ 4 = \frac{8}{2} \]
\[ 4 = 4 \]

$(3, 4)$ satisfies the equation of the altitude from $B$ through $AC$.

\[ x = 3 \quad \text{Equation of altitude from } C \text{ through } AB \]
\[ 3 = 3 \quad \text{Substitute } X(3, 4) \text{ for } x. \]

$(3, 4)$ satisfies the equation of the altitude from $C$ through $AB$.

4. State your conclusion.

$X(3, 4)$ is the orthocenter of $\triangle ABC$ with vertices $A(1, 6)$, $B(7, 6)$, and $C(3, 2)$ because $X$ satisfies each of the equations of the altitudes of the triangle.
Example 3

\( \triangle ABC \) has vertices \( A (-2, 4), B (5, 4), \) and \( C (3, -2). \) Find the equation of each median of \( \triangle ABC \) to verify that (2, 2) is the centroid of \( \triangle ABC. \)

1. Identify known information.
   
   \( \triangle ABC \) has vertices \( A (-2, 4), B (5, 4), \) and \( C (3, -2). \)

   The centroid is \( X (2, 2). \)

   The centroid of a triangle is the intersection of the medians of the triangle.
2. Determine the midpoint of each side of the triangle.

Use the midpoint formula to find the midpoint of \( \overline{AB} \).

\[
\left( \frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2} \right)
\]

Midpoint formula

\[
\left( \frac{-2 + 5}{2}, \frac{4 + 4}{2} \right)
\]

Substitute \((-2, 4)\) and \((5, 4)\) for \((x_1, y_1)\) and \((x_2, y_2)\).

\[
\left( \frac{3}{2}, \frac{8}{2} \right)
\]

Simplify.

The midpoint of \(\overline{AB}\) is \(\left( \frac{3}{2}, 4 \right)\).

Use the midpoint formula to find the midpoint of \(\overline{BC}\).

\[
\left( \frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2} \right)
\]

Midpoint formula

\[
\left( \frac{5 + 3}{2}, \frac{4 + (-2)}{2} \right)
\]

Substitute \((5, 4)\) and \((3, -2)\) for \((x_1, y_1)\) and \((x_2, y_2)\).

\[
\left( \frac{8}{2}, \frac{2}{2} \right)
\]

Simplify.

The midpoint of \(\overline{BC}\) is \((4, 1)\).  

(continued)
Use the midpoint formula to find the midpoint of $AC$.

$$
\left( \frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2} \right)
$$

Midpoint formula

$$
\left( \frac{-2 + 3}{2}, \frac{4 + (-2)}{2} \right)
$$

Substitute $(-2, 4)$ and $(3, -2)$ for $(x_1, y_1)$ and $(x_2, y_2)$.

$$
\left( \frac{1}{2}, \frac{2}{2} \right)
$$

Simplify.

$$
\left( \frac{1}{2}, 1 \right)
$$

The midpoint of $AC$ is $\left( \frac{1}{2}, 1 \right)$. 
3. Determine the medians of the triangle. 

Find the equation of \( \overline{AU} \), which is the line that passes through \( A \) and the midpoint of \( \overline{BC} \).

Use the slope formula to calculate the slope of \( \overline{AU} \).

\[
m = \frac{y_2 - y_1}{x_2 - x_1}
\]

Slope formula

\[
m = \frac{(1) - (4)}{(4) - (-2)}
\]

Substitute \((-2, 4)\) and \((4, 1)\) for \((x_1, y_1)\) and \((x_2, y_2)\).

\[
m = \frac{-3}{6}
\]

Simplify.

\[
m = -\frac{1}{2}
\]

The slope of \( \overline{AU} \) is \(-\frac{1}{2}\).

Find the \( y \)-intercept of \( \overline{AU} \).

\[
y - y_1 = m(x - x_1)
\]

Point-slope form of a line

\[
y - 4 = -\frac{1}{2}[x - (-2)]
\]

Substitute \((-2, 4)\) for \((x_1, y_1)\) and \(-\frac{1}{2}\) for \( m \).

\[
y - 4 = -\frac{1}{2}x + 1
\]

Simplify.

\[
y = \frac{1}{2}x + 3
\]

The equation of \( \overline{AU} \) that passes through \( A \) and the midpoint of \( \overline{BC} \) is \( y = -\frac{1}{2}x + 3 \).

(continued)
Find the equation of $BV$, which is the line that passes through $B$ and the midpoint of $AC$.

Use the slope formula to calculate the slope of $BV$.

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$  
Slope formula

$$m = \frac{(1) - (4)}{\left(\frac{1}{2}\right) - (5)}$$  
Substitute (5, 4) and $\left(\frac{1}{2}, 1\right)$ for $(x_1, y_1)$ and $(x_2, y_2)$.

$$m = \frac{-3}{9} = -\frac{1}{3}$$  
Simplify.

The slope of $BV$ is $\frac{2}{3}$.

Find the $y$-intercept of $BV$.

$$y - y_1 = m(x - x_1)$$  
Point-slope form of a line

$$y - 4 = -\frac{2}{3}(x - 5)$$  
Substitute (5, 4) for $(x_1, y_1)$ and $\frac{2}{3}$ for $m$.

$$y - 4 = \frac{2}{3}x - \frac{10}{3}$$  
Simplify.

$$y = -x + \frac{2}{3}$$

The equation of $BV$ that passes through $B$ and the midpoint of $AC$ is $y = -x + \frac{2}{3}$.

(continued)
Find the equation of $\overline{CT}$, which is the line that passes through $C$ and the midpoint of $\overline{AB}$.

Use the slope formula to calculate the slope of $\overline{CT}$.

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$  \hspace{1cm} \text{Slope formula}

$$m = \frac{(4) - (-2)}{(3) - (-3)}$$  \hspace{1cm} \text{Substitute (3, -2) and } \left(\frac{3}{2}, 4\right) \text{ for} \ (x_1, y_1) \text{ and } (x_2, y_2).

$$m = \frac{6}{3} = \frac{-3}{2}$$  \hspace{1cm} \text{Simplify.}

$$m = -4$$

The slope of $\overline{CT}$ is $-4$.

Find the $y$-intercept of $\overline{CT}$.

$$y - y_1 = m(x - x_1)$$  \hspace{1cm} \text{Point-slope form of a line}

$$y - (-2) = -4(x - 3)$$  \hspace{1cm} \text{Substitute (3, -2) for } (x_1, y_1) \text{ and } -4 \text{ for } m.

$$y + 2 = -4(x - 3)$$  \hspace{1cm} \text{Simplify.}

$$y + 2 = -4x + 12$$

$$y = -4x + 10$$

The equation of $\overline{CT}$ that passes through $C$ and the midpoint of $\overline{AC}$ is $y = -4x + 10$. 
4. Verify that \( X (2, 2) \) is the intersection of the three medians.

For \( (2, 2) \) to be the intersection of the three medians, the point must satisfy each of the equations: \( y = -\frac{1}{2}x + 3 \), \( y = \frac{2}{3}x + \frac{2}{3} \), and \( y = -4x + 10 \).

\[
y = -\frac{1}{2}x + 3 \quad \text{Equation of the median from } A \text{ to the midpoint of } BC
\]

\[
(2) = -\frac{1}{2}(2) + 3 \quad \text{Substitute } X (2, 2) \text{ for } (x, y).
\]

\[
2 = -1 + 3 \quad \text{Simplify.}
\]

\[
2 = 2
\]

\( (2, 2) \) satisfies the equation of the median from \( A \) to the midpoint of \( BC \).

\[
y = -\frac{2}{3}x + \frac{2}{3} \quad \text{Equation of the median from } B \text{ to the midpoint of } AC
\]

\[
(2) = \frac{2}{3}(2) + \frac{2}{3} \quad \text{Substitute } X (2, 2) \text{ for } (x, y).
\]

\[
2 = \frac{4}{3} + \frac{2}{3} \quad \text{Simplify.}
\]

\[
2 = \frac{6}{3}
\]

\[
2 = 2
\]

\( (2, 2) \) satisfies the equation of the median from \( B \) to the midpoint of \( AC \).

\[
y = -4x + 10 \quad \text{Equation of the median from } C \text{ to the midpoint of } AB
\]

\[
(2) = -4(2) + 10 \quad \text{Substitute } X (2, 2) \text{ for } (x, y).
\]

\[
(2) = -8 + 10 \quad \text{Simplify.}
\]

\[
2 = 2
\]

\( (2, 2) \) satisfies the equation of the median from \( C \) to the midpoint of \( AB \).
5. State your conclusion.

\( X (2, 2) \) is the centroid of \( \triangle ABC \) with vertices \( A (-2, 4), B (5, 4), \) and \( C (3, -2) \) because \( X \) satisfies each of the equations of the medians of the triangle.

**Example 4**

Using \( \triangle ABC \) from Example 3, which has vertices \( A (-2, 4), B (5, 4), \) and \( C (3, -2) \), verify that the centroid, \( X (2, 2) \), is \( \frac{2}{3} \) the distance from each vertex.
1. Identify the known information.

\( \triangle ABC \) has vertices \( A(-2, 4), B(5, 4), \) and \( C(3, -2) \).

The centroid is \( X(2, 2) \).

The midpoints of \( \triangle ABC \) are \( T\left(\frac{3}{2}, 4\right), U(4, 1), \) and \( V\left(\frac{1}{2}, 1\right) \).

2. Use the distance formula to show that point \( X(2, 2) \) is \( \frac{2}{3} \) the distance from each vertex.

Use the distance formula to calculate the distance from \( A \) to \( U \).

\[
d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}
\]

Distance formula

\[
d = \sqrt{((4) - (-2))^2 + (1) - (4))^2}
\]

Substitute \((-2, 4)\) and \((4, 1)\) for \((x_1, y_1)\) and \((x_2, y_2)\).

\[
d = \sqrt{(6)^2 + (-3)^2}
\]

Simplify.

\[
d = \sqrt{36 + 9} = \sqrt{45} = \sqrt{9 \cdot 5} = 3\sqrt{5}
\]

The distance from \( A \) to \( U \) is \( 3\sqrt{5} \) units.

Calculate the distance from \( X \) to \( A \).

\[
d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}
\]

Distance formula

\[
d = \sqrt{((-2) - (2))^2 + (4) - (2))^2}
\]

Substitute \((2, 2)\) and \((-2, 4)\) for \((x_1, y_1)\) and \((x_2, y_2)\).

\[
d = \sqrt{(-4)^2 + (2)^2}
\]

Simplify.

\[
d = \sqrt{16 + 4} = \sqrt{20} = \sqrt{4 \cdot 5} = 2\sqrt{5}
\]

The distance from \( X \) to \( A \) is \( 2\sqrt{5} \) units.

(continued)
\[
\frac{2}{3} AU = XA \\
\frac{2}{3}(3\sqrt{5}) = 2\sqrt{5} \\
2\sqrt{5} = 2\sqrt{5}
\]

Centroid Theorem

Substitute the distances found for \(AU\) and \(XA\).

Simplify.

\(X\) is \(\frac{2}{3}\) the distance from \(A\).

Use the distance formula to calculate the distance from \(B\) to \(V\).

\[
d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}
\]

Distance formula

\[
d = \sqrt{\left(\frac{1}{2} - 5\right)^2 + [1 - 4]^2}
\]

Substitute \((5, 4)\) and \(\left(\frac{1}{2}, 1\right)\) for \((x_1', y_1')\) and \((x_2', y_2')\).

Simplify.

\[
d = \sqrt{\left(\frac{9}{2}\right)^2 + (-3)^2}
\]

\[
d = \sqrt{\frac{81}{4} + 9}
\]

\[
d = \sqrt{\frac{117}{4} = \frac{\sqrt{9 \cdot 13}}{4} = \frac{3\sqrt{13}}{2}}
\]

The distance from \(B\) to \(V\) is \(\frac{3\sqrt{13}}{2}\) units.

Calculate the distance from \(X\) to \(B\).

\[
d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}
\]

Distance formula

\[
d = \sqrt{(5 - 2)^2 + (4 - 2)^2}
\]

Substitute \((2, 2)\) and \((5, 4)\) for \((x_1', y_1')\) and \((x_2', y_2')\).

Simplify.

\[
d = \sqrt{3^2 + 2^2}
\]

\[
d = \sqrt{9 + 4}
\]

\[
d = \sqrt{13}
\]

The distance from \(X\) to \(B\) is \(\sqrt{13}\) units.  

(continued)
\[ \frac{2}{3} BV = XB \]  
\quad \text{Centroid Theorem}

\[ \frac{2}{3} \left( \frac{3\sqrt{13}}{2} \right) = \sqrt{13} \]  
\quad \text{Substitute the distances found for } BV \text{ and } XB.

\[ \sqrt{13} = \sqrt{13} \]  
\quad \text{Simplify.}

\( X \) is \( \frac{2}{3} \) the distance from \( B \).

Use the distance formula to calculate the distance from \( C \) to \( T \).

\[
d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \quad \text{Distance formula}
\]

\[
d = \sqrt{\left[ \left( \frac{3}{2} \right) - (3) \right]^2 + [(4) - (-2)]^2} \quad \text{Substitute } (3, -2) \text{ and } \left( \frac{3}{2}, 4 \right) \text{ for } (x_1, y_1) \text{ and } (x_2, y_2).
\]

\[
d = \sqrt{\left( \frac{3}{2} \right)^2 + (6)^2} \quad \text{Simplify.}
\]

\[
d = \sqrt{\frac{9}{4} + 36}
\]

\[
d = \frac{\sqrt{153}}{4} = \frac{\sqrt{9 \cdot 17}}{\sqrt{4}} = \frac{3 \sqrt{17}}{2}
\]

The distance from \( C \) to \( T \) is \( \frac{3 \sqrt{17}}{2} \) units.  

(continued)
Calculate the distance from $X$ to $C$.

\[ d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \quad \text{Distance formula} \]

\[ d = \sqrt{(3) - (2))^2 + [(-2) - (-2))^2} \]

\[ d = \sqrt{1^2 + (-4)^2} \]

\[ d = \sqrt{1+16} \]

\[ d = \sqrt{17} \]

The distance from $X$ to $C$ is $\sqrt{17}$ units.

\[ \frac{2}{3} CT = XC \quad \text{Centroid Theorem} \]

\[ \frac{2}{3} \left( \frac{3\sqrt{17}}{2} \right) = \sqrt{17} \quad \text{ Substitute the distances found for } CT \text{ and } XC. \]

\[ \sqrt{17} = \sqrt{17} \quad \text{Simplify.} \]

$X$ is $\frac{2}{3}$ the distance from $C$.

The centroid, $X(2, 2)$, is $\frac{2}{3}$ the distance from each vertex.
Problem-Based Task 1.9.4: Sailing Centroid

When sailing, the boat must remain balanced at all times. There are many factors that can affect the stability and balance of a sailboat. One such factor is the determination of the correct center of effort, or the middle of the sail area. This center is found by determining the centroid of the sail. Donzel’s sail is a right triangle with a base of 6 feet and a height of 15 feet. The sail is represented on the graph below. Donzel has found what he thinks is the centroid of the sail, marked by point D on the graph. Has Donzel correctly identified the centroid?
Problem-Based Task 1.9.4: Sailing Centroid

Coaching

a. What is the definition of a centroid?

b. What are the vertices of the triangle representing the sail?

c. What is the midpoint of $AB$?

d. What is the midpoint of $BC$?

e. What is the midpoint of $AC$?

f. What is the equation of the median that passes through $\angle A$?

g. What is the equation of the median that passes through $\angle B$?

h. What is the equation of the median that passes through $\angle C$?

i. Does the point (4, 5) satisfy the equation of each median?

j. Has Donzel correctly identified the centroid?
Problem-Based Task 1.9.4: Sailing Centroid

Coaching Sample Responses

a. What is the definition of a centroid?
   The centroid of a triangle is the intersection of the medians of the triangle.

b. What are the vertices of the triangle representing the sail?
   The vertices of the triangle representing the sail are \(A(0, 0), B(6, 0),\) and \(C(6, 15)\).

c. What is the midpoint of \(AB\)?
   Use the midpoint formula to find the midpoint of \(AB\).
   \[
   \left( \frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2} \right)
   \]
   Midpoint formula
   \[
   \left( \frac{0 + 6}{2}, \frac{0 + 0}{2} \right)
   \]
   Substitute \((0, 0)\) and \((6, 0)\) for \((x_1, y_1)\) and \((x_2, y_2)\).
   \[
   \left( \frac{6}{2}, \frac{0}{2} \right)
   \]
   Simplify.
   \[
   (3, 0)
   \]
   The midpoint of \(AB\) is \((3, 0)\).

d. What is the midpoint of \(BC\)?
   Use the midpoint formula to find the midpoint of \(BC\).
   \[
   \left( \frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2} \right)
   \]
   Midpoint formula
   \[
   \left( \frac{6 + 6}{2}, \frac{0 + 15}{2} \right)
   \]
   Substitute \((6, 0)\) and \((6, 15)\) for \((x_1, y_1)\) and \((x_2, y_2)\).
   \[
   \left( \frac{12}{2}, \frac{15}{2} \right)
   \]
   Simplify.
   \[
   (6, 7.5)
   \]
   The midpoint of \(BC\) is \((6, 7.5)\).
e. What is the midpoint of \( \overline{AC} \) ?

Use the midpoint formula to find the midpoint of \( \overline{AC} \).

\[
\left( \frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2} \right)
\]

Midpoint formula

\[
\left( \frac{0 + 6}{2}, \frac{0 + 15}{2} \right)
\]

Substitute (0, 0) and (6, 15) for \((x_1, y_1)\) and \((x_2, y_2)\).

\[
\left( \frac{6}{2}, \frac{15}{2} \right)
\]

Simplify.

\((3, 7.5)\)

The midpoint of \( \overline{AC} \) is (3, 7.5).

f. What is the equation of the median that passes through \( A \)?

Find the equation of the line that passes through \( A \) and the midpoint of \( \overline{BC} \).

Use the slope formula to calculate the slope of this equation.

\[
m = \frac{y_2 - y_1}{x_2 - x_1}
\]

Slope formula

\[
m = \frac{7.5 - 0}{6 - 0}
\]

Substitute (0, 0) and (6, 7.5) for \((x_1, y_1)\) and \((x_2, y_2)\).

\[
m = \frac{7.5}{6}
\]

Simplify.

\[m = 1.25\]

The slope of the line that passes through \( A \) and the midpoint of \( \overline{BC} \) is 1.25.

Find the \( y \)-intercept of the line that passes through \( A \) and the midpoint of \( \overline{BC} \).

\[
y - y_1 = m(x - x_1)
\]

Point-slope form of a line

\[
y - 0 = 1.25(x - 0)
\]

Substitute (0, 0) for \((x_1, y_1)\) and 1.25 for \(m\).

\[
y = 1.25x
\]

Simplify.

The equation of the line that passes through \( A \) and the midpoint of \( \overline{BC} \) is \( y = 1.25x \).
g. What is the equation of the median that passes through $B$?

Find the equation of the line that passes through $B$ and the midpoint of $\overline{AC}$.

Use the slope formula to calculate the slope of this equation.

\[
m = \frac{y_2 - y_1}{x_2 - x_1}
\]

Slope formula

\[
m = \frac{(7.5) - (0)}{(3) - (6)}
\]

Substitute $(6, 0)$ and $(3, 7.5)$ for $(x_1, y_1)$ and $(x_2, y_2)$.

\[
m = \frac{7.5}{-3}
\]

Simplify.

\[
m = -2.5
\]

The slope of the line that passes through $B$ and the midpoint of $\overline{AC}$ is $-2.5$.

Find the $y$-intercept of the line that passes through $B$ and the midpoint of $\overline{AC}$.

\[
y - y_1 = m(x - x_1)
\]

Point-slope form of a line

\[
y - 0 = -2.5(x - 6)
\]

Substitute $(6, 0)$ for $(x_1, y_1)$ and $-2.5$ for $m$.

\[
y - 0 = -2.5x + 15
\]

Simplify.

\[
y = -2.5x + 15
\]

The equation of the line that passes through $B$ and the midpoint of $\overline{AC}$ is $y = -2.5x + 15$.

h. What is the equation of the median that passes through $C$?

Find the equation of the line that passes through $C$ and the midpoint of $\overline{AB}$.

Use the slope formula to calculate the slope of this equation.

\[
m = \frac{y_2 - y_1}{x_2 - x_1}
\]

Slope formula

\[
m = \frac{(0) - (15)}{(3) - (6)}
\]

Substitute $(6, 15)$ and $(3, 0)$ for $(x_1, y_1)$ and $(x_2, y_2)$.

\[
m = \frac{-15}{-3}
\]

Simplify.

\[
m = 5
\]

The slope of the line that passes through $C$ and the midpoint of $\overline{AB}$ is 5.
Find the $y$-intercept of the line that passes through $C$ and the midpoint of $\overline{AB}$.

$$y - y_1 = m(x - x_1)$$

Point-slope form of a line

$$y - 15 = 5(x - 6)$$

Substitute $(6, 15)$ for $(x_1, y_1)$ and $5$ for $m$.

$$y - 15 = 5x - 30$$

Simplify.

$$y = 5x - 15$$

The equation of the line that passes through $C$ and the midpoint of $\overline{AB}$ is $y = 5x - 15$.

i. Does the point $(4, 5)$ satisfy the equation of each median?

For $(4, 5)$ to be the intersection of the three medians, the point must satisfy each of the equations: $y = 1.25x$, $y = -2.5x + 15$, and $y = 5x - 15$.

$$y = 1.25x$$

Equation of the median from $A$ to the midpoint of $\overline{BC}$

$$5 = 1.25(4)$$

Substitute $(4, 5)$ for $(x, y)$.

$$5 = 5$$

$(4, 5)$ satisfies the equation of the median from $A$ to the midpoint of $\overline{BC}$.

$$y = -2.5x + 15$$

Equation of the median from $B$ to the midpoint of $\overline{AC}$

$$5 = -2.5(4) + 15$$

Substitute $(4, 5)$ for $(x, y)$.

$$5 = -10 + 15$$

Simplify.

$$5 = 5$$

$(4, 5)$ satisfies the equation of the median from $B$ to the midpoint of $\overline{AC}$.

$$y = 5x - 15$$

Equation of the median from $C$ to the midpoint of $\overline{AB}$

$$5 = 5(4) - 15$$

Substitute $(4, 5)$ for $(x, y)$.

$$5 = 20 - 10$$

Simplify.

$$5 = 5$$

$(4, 5)$ satisfies the equation of the median from $C$ to the midpoint of $\overline{AB}$. 
j. Has Donzel correctly identified the centroid?

Yes; $D (4, 5)$ is the centroid of $\triangle ABC$ with vertices $A (0, 0)$, $B (3, 0)$, and $C (6, 15)$ because $(4, 5)$ satisfies each of the equations of the medians of the triangle.

**Recommended Closure Activity**

Select one or more of the essential questions for a class discussion or as a journal entry prompt.
Practice 1.9.4: Proving Centers of Triangles

Use what you know about centers of triangles to complete each problem.

1. \( \triangle ABC \) has vertices \( A (0, 0), B (0, -8), \) and \( C (4, -8) \). Justify that \( (2, -4) \) is the circumcenter of \( \triangle ABC \).

2. \( \triangle ABC \) has vertices \( A (-4, 10), B (8, -2), \) and \( C (12, 10) \). Justify that \( (8, 6) \) is the orthocenter of \( \triangle ABC \).

3. \( \triangle ABC \) has vertices \( A (6, 9), B (0, 0), \) and \( C (-15, 0) \). Justify that \( (-3, 3) \) is the centroid of \( \triangle ABC \).

4. Verify that the centroid, \( (-3, 3) \), of \( \triangle ABC \) with vertices \( A (6, 9), B (0, 0), \) and \( C (-15, 0) \) is \( \frac{2}{3} \) the distance from each vertex to the midpoint of the opposite side.

5. \( \triangle ABC \) has vertices \( A (-3, 2), B (4, 7), \) and \( C (0, 6) \). Will the incenter be inside, outside, or on a side of \( \triangle ABC \)? Explain your answer.

6. \( \triangle ABC \) has vertices \( A (-4, 6), B (1, 6), \) and \( C (4, 9) \). Will the orthocenter be inside, outside, or on a side of \( \triangle ABC \)? Explain your answer.

continued
7. The Incenter Theorem states the incenter of a triangle is equidistant from the sides of a triangle. Prove this theorem using the information below.

Given: \( \triangle ABC \) has angle bisectors \( AN, BP, \) and \( CM \).

\( XR \perp AB, XS \perp BC, \) and \( XT \perp AC \).

Prove: \( XR = XS = XT \)
8. A fire station is to be built to assist three towns. The relative locations of the towns and the concurrent roads connecting the towns are shown below. If the fire station cannot be built outside the area of the triangle, which point(s) of concurrency cannot be used to determine the location of the fire station?

9. A circular pond is to be constructed in a triangular park. Which center of the park should be determined to create the largest possible pond? Explain your answer.

10. The park’s recreation director is determining the location of a new water fountain to be equidistant from the swings, basketball court, and gazebo. Which center of the triangle created between each location should be determined? Explain your answer.
Lesson 10: Proving Theorems About Parallelograms

Common Core Georgia Performance Standard
MCC9–12.G.CO.11

Essential Questions
1. What makes a quadrilateral a parallelogram?
2. What is the hierarchy of quadrilaterals?
3. What theorems are used for parallelograms?
4. How are rectangles, rhombuses, squares, kites, and trapezoids alike and different?

WORDS TO KNOW
concave polygon a polygon with at least one interior angle greater than $180^\circ$ and at least one diagonal that does not lie entirely inside the polygon
consecutive angles angles that lie on the same side of a figure
convex polygon a polygon with no interior angle greater than $180^\circ$; all diagonals lie inside the polygon
diagonal a line that connects nonconsecutive vertices
isosceles trapezoid a trapezoid with one pair of opposite parallel lines and congruent legs
kite a quadrilateral with two distinct pairs of congruent sides that are adjacent
parallelogram a special type of quadrilateral with two pairs of opposite sides that are parallel; denoted by the symbol $\square$
quadrilateral a polygon with four sides
rectangle a special parallelogram with four right angles
rhombus a special parallelogram with all four sides congruent
square a special parallelogram with four congruent sides and four right angles
trapezoid a quadrilateral with exactly one pair of opposite parallel lines
UNIT 1 • SIMILARITY, CONGRUENCE, AND PROOFS
Lesson 10: Proving Theorems About Parallelograms

Recommended Resources

  http://www.walch.com/rr/00037

  Review different quadrilaterals with this interactive site. Click the name of a
  quadrilateral to view its shape and definition, then select and drag the vertices to
  change or rotate the shape. Options to view the angle measures and/or diagonals of
  each given quadrilateral are also included.

- Math Warehouse. “Parallelograms.”
  http://www.walch.com/rr/00038

  This website gives a brief overview of the properties of parallelograms. Users can
  examine the relationships among sides and angles with an interactive parallelogram.
  Each section offers three test questions, with answers provided when you click on the
  “Answer” button.

- Math Warehouse. “Rhombus: Properties and Shape.”
  http://www.walch.com/rr/00039

  Rhombuses are defined and explained with examples at this site. Clickable practice
  questions are provided to test understanding.

  Rhombuses, Squares.”
  http://www.walch.com/rr/00040

  This site provides a simple lesson on the properties of rectangles, rhombuses, and
  squares, with concise descriptions and examples.
Lesson 1.10.1: Proving Properties of Parallelograms

Warm-Up 1.10.1

Parking lots have standard measures for the width of each space, the backout distance required, and the angle measure. There are several acceptable angles for parking spaces. The desired angle is marked as $\angle 1$ in the illustration below.

1. Find the measure of $\angle 2$ given the desired angle. Assume that the width of the parking spaces is constant, meaning the lines of the spaces are parallel.

<table>
<thead>
<tr>
<th>$m\angle 1$</th>
<th>$m\angle 2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>45$^\circ$</td>
<td></td>
</tr>
<tr>
<td>60$^\circ$</td>
<td></td>
</tr>
<tr>
<td>75$^\circ$</td>
<td></td>
</tr>
<tr>
<td>90$^\circ$</td>
<td></td>
</tr>
</tbody>
</table>

2. Explain how you determined your angle measures.
Lesson 1.10.1: Proving Properties of Parallelograms

Common Core Georgia Performance Standard

MCC9–12.G.CO.11

Warm-Up 1.10.1 Debrief

Parking lots have standard measures for the width of each space, the backout distance required, and the angle measure. There are several acceptable angles for parking spaces. The desired angle is marked as $\angle 1$ in the illustration below.

1. Find the measure of $\angle 2$ given the desired angle. Assume that the width of the parking spaces is constant, meaning the lines of the spaces are parallel.

<table>
<thead>
<tr>
<th>$m\angle 1$</th>
<th>$m\angle 2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>45°</td>
<td>135°</td>
</tr>
<tr>
<td>60°</td>
<td>120°</td>
</tr>
<tr>
<td>75°</td>
<td>105°</td>
</tr>
<tr>
<td>90°</td>
<td>90°</td>
</tr>
</tbody>
</table>
2. Explain how you determined your angle measures.

\( \angle 1 \) and \( \angle 2 \) are same-side interior angles. Same-side interior angles are supplementary. Therefore, use the equation as follows:

\[
m \angle 1 + m \angle 2 = 180
\]

\[
m \angle 2 = 180 - m \angle 1
\]

Substitute each given measure of \( \angle 1 \) into the equation in order to solve for the measure of \( \angle 2 \).

**Connection to the Lesson**

- Students will be using supplementary angles when finding measures of consecutive angles in a parallelogram.
- Students will extend their understanding of parallel lines intersected by a transversal to two sets of parallel lines that ultimately intersect to form a parallelogram.
- Students will further investigate the relationships of these angles in two intersecting sets of parallel lines.
Introduction

What does it mean to be opposite? What does it mean to be consecutive? Think about a rectangular room. If you put your back against one corner of that room and looked directly across the room, you would be looking at the opposite corner. If you looked to your right, that corner would be a consecutive corner. If you looked to your left, that corner would also be a consecutive corner. The walls of the room could also be described similarly. If you were to stand with your back at the center of one wall, the wall straight across from you would be the opposite wall. The walls next to you would be consecutive walls. There are two pairs of opposite walls in a rectangular room, and there are two pairs of opposite angles. Before looking at the properties of parallelograms, it is important to understand what the terms opposite and consecutive mean.

Key Concepts

- A **quadrilateral** is a polygon with four sides.
- A **convex polygon** is a polygon with no interior angle greater than 180° and all diagonals lie inside the polygon.
- A **diagonal** of a polygon is a line that connects nonconsecutive vertices.
Convex polygons are contrasted with concave polygons.

A concave polygon is a polygon with at least one interior angle greater than 180° and at least one diagonal that does not lie entirely inside the polygon.

A parallelogram is a special type of quadrilateral with two pairs of opposite sides that are parallel.

By definition, if a quadrilateral has two pairs of opposite sides that are parallel, then the quadrilateral is a parallelogram.

Parallelograms are denoted by the symbol \( \square \).

If a polygon is a parallelogram, there are five theorems associated with it.
• In a parallelogram, both pairs of opposite sides are congruent.

**Theorem**

If a quadrilateral is a parallelogram, opposite sides are congruent.

\[ \overline{AB} \cong \overline{DC} \]
\[ \overline{AD} \cong \overline{BC} \]

The converse is also true. If the opposite sides of a quadrilateral are congruent, then the quadrilateral is a parallelogram.
Parallelograms also have two pairs of opposite angles that are congruent.

**Theorem**

If a quadrilateral is a parallelogram, opposite angles are congruent.

\[ \angle A \cong \angle C \]
\[ \angle B \cong \angle D \]

The converse is also true. If the opposite angles of a quadrilateral are congruent, then the quadrilateral is a parallelogram.
Consecutive angles are angles that lie on the same side of a figure.

In a parallelogram, consecutive angles are supplementary; that is, they sum to 180°.

**Theorem**

If a quadrilateral is a parallelogram, then consecutive angles are supplementary.

\[
\begin{align*}
\angle A + \angle B &= 180 \\
\angle B + \angle C &= 180 \\
\angle C + \angle D &= 180 \\
\angle D + \angle A &= 180
\end{align*}
\]
The diagonals of a parallelogram have a relationship. They bisect each other.

**Theorem**

The diagonals of a parallelogram bisect each other.

\[ \overline{AP} \cong \overline{PC} \]
\[ \overline{BP} \cong \overline{PD} \]

The converse is also true. If the diagonals of a quadrilateral bisect each other, then the quadrilateral is a parallelogram.
Notice that each diagonal divides the parallelogram into two triangles. Those two triangles are congruent.

Theorem
The diagonal of a parallelogram forms two congruent triangles.

\[ \triangle BAD \cong \triangle DCB \]

Common Errors/Misconceptions
- thinking that all angles in a parallelogram are congruent even if the parallelogram isn’t a rectangle or square
- misidentifying opposite pairs of sides
- misidentifying opposite pairs of angles and consecutive angles
Guided Practice 1.10.1

Example 1

Quadrilateral $ABCD$ has the following vertices: $A (-4, 4), B (2, 8), C (3, 4),$ and $D (-3, 0)$. Determine whether the quadrilateral is a parallelogram. Verify your answer using slope and distance to prove or disprove that opposite sides are parallel and opposite sides are congruent.

1. Graph the figure.
UNIT 1 • SIMILARITY, CONGRUENCE, AND PROOFS
Lesson 10: Proving Theorems About Parallelograms

2. Determine whether opposite pairs of lines are parallel.

Calculate the slope of each line segment.

\[
\begin{align*}
AB & = (8 - 4) \quad (4 - 8) \quad -4 \\
\Delta x & = 2 - (-4) \quad 6 \quad 3 \\
\Delta y & = 4 \\
m_{AB} & = \frac{\Delta y}{\Delta x} = \frac{4}{6} = \frac{2}{3} \\
BC & = (4 - 8) \quad -4 \\
\Delta x & = 3 - 2 \quad 1 \\
\Delta y & = 6 \\
m_{BC} & = \frac{\Delta y}{\Delta x} = \frac{-4}{1} \\
DC & = (4 - 0) \quad 2 \\
\Delta x & = 3 - (-3) \quad 6 \quad 3 \\
\Delta y & = 4 \\
m_{DC} & = \frac{\Delta y}{\Delta x} = \frac{2}{3} \\
AD & = (0 - 4) \quad -4 \\
\Delta x & = [-3 - (-4)] \quad 1 \\
\Delta y & = 6 \\
m_{AD} & = \frac{\Delta y}{\Delta x} = \frac{-4}{1} = -4
\end{align*}
\]

Calculating the slopes, we can see that the opposite sides are parallel because the slopes of the opposite sides are equal. By the definition of a parallelogram, quadrilateral \(ABCD\) is a parallelogram.

3. Verify that the opposite sides are congruent.

Calculate the distance of each segment using the distance formula.

\[
\begin{align*}
AB & = \sqrt{(2 - (-4))^2 + (8 - 4)^2} \\
& = \sqrt{36 + 16} \\
& = \sqrt{52} \\
& = 2\sqrt{13} \\
BC & = \sqrt{(3 - 2)^2 + (4 - 8)^2} \\
& = \sqrt{1 + 16} \\
& = \sqrt{17} \\
\end{align*}
\]

\[
\begin{align*}
AB & = \sqrt{(6)^2 + (4)^2} \\
BC & = \sqrt{(1)^2 + (-4)^2} \\
& = \sqrt{1 + 16} \\
& = \sqrt{17} \\
\end{align*}
\]

\[
\begin{align*}
DC & = \sqrt{[3 - (-3)]^2 + (4 - 0)^2} \\
AD & = \sqrt{[-3 - (-4)]^2 + (0 - 4)^2} \\
DC & = \sqrt{36 + 16} \\
AD & = \sqrt{1 + 16} \\
DC & = \sqrt{52} = 2\sqrt{13} \\
AD & = \sqrt{17}
\end{align*}
\]

From the distance formula, we can see that opposite sides are congruent. Because of the definition of congruence and since \(AB = DC\) and \(BC = AD\), then \(AB \cong DC\) and \(BC \cong AD\).
Example 2

Use the parallelogram from Example 1 to verify that the opposite angles in a parallelogram are congruent and consecutive angles are supplementary given that $\overrightarrow{AD} \parallel \overrightarrow{BC}$ and $\overrightarrow{AB} \parallel \overrightarrow{DC}$.
1. Extend the lines in the parallelogram to show two pairs of intersecting lines and label the angles with numbers.

2. Prove $\angle 4 \cong \angle 9$.

   $\overline{AD} \parallel \overline{BC}$ and $\overline{AB} \parallel \overline{DC}$  \hspace{1cm} \text{Given}

   $\angle 4 \cong \angle 13$  \hspace{1cm} \text{Alternate Interior Angles Theorem}

   $\angle 13 \cong \angle 16$  \hspace{1cm} \text{Vertical Angles Theorem}

   $\angle 16 \cong \angle 9$  \hspace{1cm} \text{Alternate Interior Angles Theorem}

   $\angle 4 \cong \angle 9$  \hspace{1cm} \text{Transitive Property}

We have proven that one pair of opposite angles in a parallelogram is congruent.
3. Prove $\angle 7 \cong \angle 14$.

\[
AD \parallel BC \text{ and } AB \parallel DC \quad \text{Given}
\]

$\angle 7 \cong \angle 10$ \hspace{1cm} Alternate Interior Angles Theorem

$\angle 10 \cong \angle 11$ \hspace{1cm} Vertical Angles Theorem

$\angle 11 \cong \angle 14$ \hspace{1cm} Alternate Interior Angles Theorem

$\angle 7 \cong \angle 14$ \hspace{1cm} Transitive Property

We have proven that both pairs of opposite angles in a parallelogram are congruent.

4. Prove that consecutive angles of a parallelogram are supplementary.

\[
AD \parallel BC \text{ and } AB \parallel DC \quad \text{Given}
\]

$\angle 4$ and $\angle 14$ are supplementary. \hspace{1cm} Same-Side Interior Angles Theorem

$\angle 14$ and $\angle 9$ are supplementary. \hspace{1cm} Same-Side Interior Angles Theorem

$\angle 9$ and $\angle 7$ are supplementary. \hspace{1cm} Same-Side Interior Angles Theorem

$\angle 7$ and $\angle 4$ are supplementary. \hspace{1cm} Same-Side Interior Angles Theorem

We have proven consecutive angles in a parallelogram are supplementary using the Same-Side Interior Angles Theorem of a set of parallel lines intersected by a transversal.
Example 3

Use the parallelogram from Example 1 to prove that diagonals of a parallelogram bisect each other.

1. Find the midpoint of $\overline{AC}$, where $M$ stands for midpoint.

   By definition, the midpoint is the point on a segment that divides the segment into two congruent parts.

   \[
   M = \left( \frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2} \right)
   \]

   Midpoint formula

   \[
   M_{\overline{AC}} = \left( \frac{-4 + 3}{2}, \frac{4 + 4}{2} \right) = \left( \frac{-1}{2}, \frac{8}{2} \right) = \left( -\frac{1}{2}, 4 \right)
   \]

   Substitute values for $x_1, x_2, y_1,$ and $y_2$, then solve.
2. Find the midpoint of $DB$.

$$M = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$$

Midpoint formula

$$M_{DB} = \left(\frac{-3 + 2}{2}, \frac{0 + 8}{2}\right) = \left(\frac{-1}{2}, \frac{8}{2}\right) = \left(-\frac{1}{2}, 4\right)$$

Substitute values for $x_1, x_2, y_1,$ and $y_2$, then solve.

3. Mark the midpoint of each segment on the graph.

Notice that the midpoint of $AC$ and the midpoint of $DB$ are the same point.

4. Write statements that prove the diagonals bisect each other.

Since $M$ is the midpoint of $AC$, $AM \cong MC$. $M$ is also a point on $DB$. Therefore, $DB$ is the bisector of $AC$. The midpoint of $DB$ is $M$. This means that $DM \cong DB$. Since $M$ is a point on $AC$, $AC$ is the bisector of $DB$. The diagonals bisect each other.
Example 4

Use the parallelogram from Example 1 and the diagonal \( \overline{DB} \) to prove that a diagonal of a parallelogram separates the parallelogram into two congruent triangles.
1. Use theorems about parallelograms to mark congruent sides.

Opposite sides of a parallelogram are congruent, as proven in Example 1.

\[ AB \cong DC \quad \text{and} \quad BC \cong AD \]

Opposite sides of a parallelogram are congruent.

So far, we know that the triangles each have two sides that are congruent to the corresponding sides of the other triangle. To prove triangles congruent, we could use ASA, SAS, or SSS. From the information we have, we could either try to find the third side congruent or the included angles congruent.
2. Use the Reflexive Property to identify a third side of the triangle that is congruent.

\[ \overline{DB} \cong \overline{DB} \] by the Reflexive Property.

Now, all three sides of the triangles are congruent.

3. State the congruent triangles.

Using SSS, we verified that \( \triangle DAB \cong \triangle BCD \). Therefore, the diagonal splits the parallelogram into two congruent triangles.
Problem-Based Task 1.10.1: Measuring Bridge Cable Tension

A bridge truss is in the shape of a parallelogram as pictured below. To support the truss, steel cables are placed on each of the diagonals of the parallelogram. Engineers are concerned with the tension in one pair of cables, and want to put a strain gauge in the center of the two cables to measure the tension. If the tension is too high, the cables could snap. How will the engineers determine where to place the strain gauges without measuring the length of each cable? Prove that the engineers’ decision will work. Use the diagram below for reference in your proof.
Problem-Based Task 1.10.1: Measuring Bridge Cable Tension

Coaching

a. What is the property of the diagonals of a parallelogram?

b. How can you use this property?

c. Which lines are parallel?

d. What are the transversals?

e. What angles are congruent and how do you know?

f. What sides are congruent and how do you know?

g. What triangles are congruent and how do you know?

h. What segments are congruent that you have not already identified? How do you know?

i. Summarize your findings and write your proof.
Problem-Based Task 1.10.1: Measuring Bridge Cable Tension

Coaching Sample Responses

a. What is the property of the diagonals of a parallelogram?
   The diagonals of a parallelogram bisect each other.

b. How can you use this property?
   Where the cables intersect is the midpoint of each cable and, therefore, the center of the cable. That is where the strain gauge should be located.

c. Which lines are parallel?
   Since the truss $ABCD$ is a parallelogram, opposite sides are parallel. This means that the top is parallel to the bottom and the left side is parallel to the right side; or, stated mathematically, $AB \parallel DC$ and $AD \parallel BC$.

d. What are the transversals?
   Each cable acts as a transversal to both sets of parallel lines. This means that $AC$ is a transversal of $AB$ and $DC$, as well as a transversal of $AD$ and $BC$. This also means that $DB$ is a transversal of $AB$ and $DC$, as well as a transversal of $AD$ and $BC$.

e. What angles are congruent and how do you know?
   Because alternate interior angles of a set of parallel lines intersected by a transversal are congruent, $\angle BAC \cong \angle DCA$, $\angle BDC \cong \angle DBA$, $\angle DAC \cong \angle ACB$, and $\angle ADB \cong \angle DBC$.
f. What sides are congruent and how do you know?

Opposite sides of a parallelogram are congruent. Therefore, $AB \cong DC$ and $AD \cong BC$.

![Diagram of a parallelogram with diagonals intersecting at a point labeled P]

g. What triangles are congruent and how do you know?

Label the intersection point of the diagonals with the letter $P$.

By SAS, $\triangle ABP \cong \triangle CDP$ and $\triangle ADP \cong \triangle CBP$.

![Diagram with triangles labeled and intersection point P]

h. What segments are congruent that you have not already identified? How do you know?

Because of CPCTC, $AP \cong CP$ and $DP \cong BP$. By the definition of bisection, $AC$ bisects $DB$ and $DB$ bisects $AC$.
i. Summarize your findings and write your proof.

Given that $ABCD$ is a parallelogram, opposite sides are parallel, so $AB \parallel DC$ and $AD \parallel BC$. Since alternate interior angles of a set of parallel lines intersected by a transversal are congruent, $\angle BAC \equiv \angle DCA$, $\angle BDC \equiv \angle DBA$, $\angle DAC \equiv \angle ACB$, and $\angle ADB \equiv \angle DBC$. Opposite sides of a parallelogram are congruent. Therefore, $AB \equiv DC$ and $AD \equiv BC$. By SAS, $\triangle ABP \cong \triangle CDP$ and $\triangle ADP \cong \triangle CBP$. Since corresponding parts of congruent triangles are congruent, $AP \equiv CP$ and $DP \equiv BP$. By the definition of bisection, $AC$ bisects $DB$ and $DB$ bisects $AC$.

**Recommended Closure Activity**

Select one or more of the essential questions for a class discussion or as a journal entry prompt.
Use slope to determine whether the given vertices form a parallelogram.

1. \(A(-2, -1), B(-1, 3), C(4, 3), \) and \(D(3, -1)\)

2. \(F(-1, 1), G(1, 3), H(4, -2), \) and \(I(2, -4)\)

Use the distance formula to determine whether the given vertices form a parallelogram.

3. \(J(0, 0), K(1, 5), L(4, 6), \) and \(M(3, 0)\)

4. \(M(0, 8), N(-1, 2), O(5, 6), \) and \(P(6, 12)\)

Use the midpoint formula to determine whether the given vertices form a parallelogram.

5. \(P(-2, 2), Q(1, 8), R(4, 5), \) and \(S(3, -2)\)

6. \(D(2, 2), E(4, 5), F(10, 6), \) and \(G(8, 3)\)
Determine the unknown angle measures and the values of $x$ and $y$ that make quadrilateral $ABCD$ a parallelogram.

7.

8.

continued
9. Given \( \square ABCD \), prove that \( \triangle DPA \cong \triangle BPC \).

10. Given \( \square ABCD \), \( \square EBHG \), and \( \square FIJG \) prove \( \angle D \cong \angle I \).
Lesson 10: Proving Theorems About Parallelograms

Lesson 1.10.2: Proving Properties of Special Quadrilaterals

Warm-Up 1.10.2

You are taking a road trip from Carrollton, Georgia, to Campton, Georgia, traveling through Atlanta. Your best efforts to avoid traffic will take you around the city of Atlanta, but you want to know how far it is “as the crow flies” (moving in a straight line) between Carrollton and Campton.

1. When looking at a map with a grid, if Carrollton lies at (2, 2) and Campton lies at (10, 4.5), calculate the distance between those two cities if 1 unit on the grid is approximately 10 miles.

2. If Atlanta is the midpoint between the two cities, where does Atlanta lie on the grid?

3. What is the distance between Atlanta and each of the cities?

4. What is the slope of the line that passes through these three cities?

5. What is the slope of a line perpendicular to the line through the cities?
Lesson 1.10.2: Proving Properties of Special Quadrilaterals

Common Core Georgia Performance Standard

MCC9–12.G.CO.11

Warm-Up 1.10.2 Debrief

You are taking a road trip from Carrollton, Georgia, to Campton, Georgia, traveling through Atlanta. Your best efforts to avoid traffic will take you around the city of Atlanta, but you want to know how far it is “as the crow flies” (moving in a straight line) between Carrollton and Campton.

1. When looking at a map with a grid, if Carrollton lies at (2, 2) and Campton lies at (10, 4.5), calculate the distance between those two cities if 1 unit on the grid is approximately 10 miles.

   Use the distance formula: \( d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \).

   \[
   d = \sqrt{(10 - 2)^2 + (4.5 - 2)^2} \\
   d = \sqrt{(8)^2 + (2.5)^2} \\
   d = \sqrt{64 + 6.25} \\
   d = \sqrt{70.25} \\
   d = 8.4
   \]

   The distance between Carrollton and Campton is about 8.4 units. Multiply that by 10 miles.

   The distance between Carrollton and Campton is about 84 miles.

2. If Atlanta is the midpoint between the two cities, where does Atlanta lie on the grid?

   Use the midpoint formula to find Atlanta’s location on the grid.

   \[
   M = \left( \frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2} \right)
   \]

   \[
   M = \left( \frac{2 + 10}{2}, \frac{2 + 4.5}{2} \right) \\
   M = \left( \frac{12}{2}, \frac{6.5}{2} \right) = (6, 3.25)
   \]

   Atlanta is at (6, 3.25) on the grid.
3. What is the distance between Atlanta and each of the cities?

By the definition of a midpoint, each shorter segment is half the distance of the total segment.

\[
\frac{84}{2} = 42 \text{ miles}
\]

Each city is about 42 miles from Atlanta.

4. What is the slope of the line that passes through these three cities?

Using the endpoints, calculate the slope of the line.

The endpoints are the grid locations of Carrollton and Campton: (2, 2) and (10, 4.5).

\[
m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{4.5 - 2}{10 - 2} = \frac{2.5}{8} = \frac{5}{16} = \frac{5}{2} \cdot \frac{1}{8} = \frac{5}{16}
\]

The slope is \(\frac{5}{16}\).

5. What is the slope of a line perpendicular to the line through the cities?

Perpendicular lines have opposite reciprocal slopes.

Since the slope of the original line is \(\frac{5}{16}\), a line perpendicular to that line would have a slope of \(-\frac{16}{5}\).

Connection to the Lesson

- Students will be using distance, midpoints, and slopes to prove properties of special quadrilaterals.
- Students will also be using distance, midpoints, and slopes to classify special quadrilaterals.
Introduction

There are many kinds of quadrilaterals. Some quadrilaterals are parallelograms; some are not. For example, trapezoids and kites are special quadrilaterals, but they are not parallelograms.

Some parallelograms are known as special parallelograms. What makes a parallelogram a more specialized parallelogram? Rectangles, rhombuses, and squares are all special parallelograms with special properties. They have all the same characteristics that parallelograms have, plus more.

Key Concepts

- A **rectangle** has four sides and four right angles.
- A rectangle is a parallelogram, so opposite sides are parallel, opposite angles are congruent, and consecutive angles are supplementary.
- The diagonals of a rectangle bisect each other and are also congruent.

### Theorem

If a parallelogram is a rectangle, then the diagonals are congruent.

\[ AC \cong DB \]
A **rhombus** is a special parallelogram with all four sides congruent. Since a rhombus is a parallelogram, opposite sides are parallel, opposite angles are congruent, and consecutive angles are supplementary. The diagonals bisect each other; additionally, they also bisect the opposite pairs of angles within the rhombus.

### Theorem
If a parallelogram is a rhombus, the diagonals of the rhombus bisect the opposite pairs of angles.

\[ \angle BAC \cong \angle CAD \cong \angle BCA \cong \angle DCA \]
\[ \angle CBD \cong \angle ABD \cong \angle ADB \cong \angle CDB \]

The diagonals of a rhombus also form four right angles where they intersect.

### Theorem
If a parallelogram is a rhombus, the diagonals are perpendicular.

\[ \overline{BD} \perp \overline{AC} \]

The converse is also true. If the diagonals of a parallelogram intersect at a right angle, then the parallelogram is a rhombus.
• A square has all the properties of a rectangle and a rhombus.
• Squares have four congruent sides and four right angles.
• The diagonals of a square bisect each other, are congruent, and bisect opposite pairs of angles.
• The diagonals are also perpendicular.

Properties of Squares

\[
\begin{align*}
\overline{AB} & \cong \overline{BC} \cong \overline{CD} \cong \overline{DA} \\
\overline{BD} & \cong \overline{AC} \\
\overline{BD} & \perp \overline{AC} \\
m \angle A & = m \angle B = m \angle C = m \angle D = 90
\end{align*}
\]
Trapezoids are quadrilaterals with exactly one pair of opposite parallel lines.

- Trapezoids are not parallelograms because they do not have two pairs of opposite lines that are parallel.
- The lines in a trapezoid that are parallel are called the bases, and the lines that are not parallel are called the legs.

Properties of Trapezoids

\[ BA \quad \text{and} \quad CD \quad \text{are the legs.} \]
\[ BC \quad \text{and} \quad AD \quad \text{are the bases.} \]
\[ BC \parallel AD \]
• **Isosceles trapezoids** have one pair of opposite parallel lines. The legs are congruent.

• Since the legs are congruent, both pairs of base angles are also congruent, similar to the legs and base angles in an isosceles triangle.

• The diagonals of an isosceles trapezoid are congruent.

![Properties of Isosceles Trapezoids](image-url)

\[
\begin{align*}
BA \quad \text{and} \quad CD & \quad \text{are the legs.} \\
BC \quad \text{and} \quad AD & \quad \text{are the bases.} \\
BC \parallel AD \\
BA \cong CD \quad \text{and} \quad AC \cong BD \\
\angle BAD \cong \angle ADC \quad \text{and} \quad \angle ABC \cong \angle BCD
\end{align*}
\]
• A **kite** is a quadrilateral with two distinct pairs of congruent sides that are adjacent.

• Kites are not parallelograms because opposite sides are not parallel.

• The diagonals of a kite are perpendicular.

Properties of Kites

<table>
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<th>Properties of Kites</th>
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A kite $ABCD$ with:

- $CD \cong CB$ and $AB \cong AD$
- $CA \perp BD$

• Quadrilaterals can be grouped according to their properties. This kind of grouping is called a hierarchy.

• In the following hierarchy of quadrilaterals, you can see that all quadrilaterals are polygons but that not all polygons are quadrilaterals.

• The arrows connecting the types of quadrilaterals indicate a special version of the category above each quadrilateral type. For example, parallelograms are special quadrilaterals. Rectangles and rhombuses are special parallelograms, and squares have all the properties of rectangles and rhombuses.
Lesson 10: Proving Theorems About Parallelograms

Hierarchy of Quadrilaterals

Common Errors/Misconceptions
- assuming both pairs of opposite sides are parallel after determining that one pair of opposite sides is parallel
- mistakenly classifying a rhombus as a square since all the sides are congruent
- confusing the properties among the special quadrilaterals
- not understanding that rectangles have the same properties as a parallelogram plus additional properties
- not understanding the hierarchy of quadrilaterals
Guided Practice 1.10.2

Example 1

Quadrilateral $ABCD$ has vertices $A (–6, 8)$, $B (2, 2)$, $C (–1, –2)$, and $D (–9, 4)$. Using slope, distance, and/or midpoints, classify $\square ABCD$ as a rectangle, rhombus, square, trapezoid, isosceles trapezoid, or kite.

1. Graph the quadrilateral.
2. Calculate the slopes of the sides to determine if opposite sides are parallel.

If opposite sides are parallel, the quadrilateral is a parallelogram.

\[
m_{AB} = \frac{\Delta y}{\Delta x} = \frac{2-8}{2-(-6)} = \frac{-6}{8} = -\frac{3}{4}
\]

\[
m_{DC} = \frac{\Delta y}{\Delta x} = \frac{2-4}{2-(-9)} = \frac{-6}{8} = -\frac{3}{4}
\]

The first pair of opposite sides is parallel: \(AB \parallel DC\).

\[
m_{AD} = \frac{\Delta y}{\Delta x} = \frac{4-8}{-9-(-6)} = \frac{-4}{-3} = \frac{4}{3}
\]

\[
m_{BC} = \frac{\Delta y}{\Delta x} = \frac{-2-2}{-1-(-2)} = \frac{-4}{-3} = \frac{4}{3}
\]

The second pair of opposite sides is parallel: \(AD \parallel BC\). Therefore, the quadrilateral is a parallelogram.
3. Examine the slopes of the consecutive sides to determine if they intersect at right angles.

If the slopes are opposite reciprocals, the lines are perpendicular and therefore form right angles. If there are four right angles, the quadrilateral is a rectangle or a square.

\[
m_{AB} = m_{DC} = \frac{-3}{4}
\]

\[
m_{AD} = m_{BC} = \frac{4}{3}
\]

\[-\frac{3}{4}\] is the opposite reciprocal of \[\frac{4}{3}\].

The slopes of the consecutive sides are perpendicular: \(AB \perp AD\) and \(DC \perp BC\). There are four right angles at the vertices. The parallelogram is a rectangle or a square.

You could also determine if the diagonals are congruent by calculating the length of each diagonal using the distance formula, \(d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}\). If the diagonals are congruent, then the parallelogram is a rectangle or square.

\[
AC = \sqrt{(-1-(-6))^2 + (-2-8)^2} \quad DB = \sqrt{[2-(-9)]^2 + (2-4)^2}
\]

\[
AC = \sqrt{(5)^2 + (-10)^2} \quad DB = \sqrt{(11)^2 + (-2)^2}
\]

\[
AC = \sqrt{25+100} \quad DB = \sqrt{121+4}
\]

\[
AC = \sqrt{125} \quad DB = \sqrt{125}
\]

\[
AC = 5\sqrt{5} \quad DB = 5\sqrt{5}
\]

The diagonals are congruent: \(AC \cong DB\). The parallelogram is a rectangle.
4. Calculate the lengths of the sides.

If all sides are congruent, the parallelogram is a rhombus or a square. Since we established that the angles are right angles, the rectangle can be more precisely classified as a square if the sides are congruent. If the sides are not congruent, the parallelogram is a rectangle.

Use the distance formula to calculate the lengths of the sides.

\[ d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \]

\[ AB = \sqrt{(2 - (-6))^2 + (2 - 8)^2} \quad AD = \sqrt{[-9 - (-6)]^2 + (4 - 8)^2} \]
\[ AB = \sqrt{(8)^2 + (-6)^2} \quad AD = \sqrt{(3)^2 + (-4)^2} \]
\[ AB = \sqrt{64 + 36} \quad AD = \sqrt{9 + 16} \]
\[ AB = 10 \quad AD = 5 \]

\[ DC = \sqrt{[-1 - (-9)]^2 + (-2 - 4)^2} \quad BC = \sqrt{(-1 - 2)^2 + (-2 - 2)^2} \]
\[ DC = \sqrt{(8)^2 + (-6)^2} \quad BC = \sqrt{(3)^2 + (-4)^2} \]
\[ DC = \sqrt{64 + 36} \quad BC = \sqrt{9 + 16} \]
\[ DC = 10 \quad BC = 5 \]

Opposite sides are congruent, which is consistent with a parallelogram, but all sides are not congruent.

5. Summarize your findings.

The quadrilateral has opposite sides that are parallel and four right angles, but not four congruent sides. This makes the quadrilateral a parallelogram and a rectangle.
Example 2
Quadrilateral $ABCD$ has vertices $A(0, 8)$, $B(11, 1)$, $C(0, –6)$, and $D(–11, 1)$. Using slope, distance, and/or midpoints, classify $\square ABCD$ as a rectangle, rhombus, square, trapezoid, isosceles trapezoid, or kite.

1. Graph the quadrilateral.

2. Calculate the slopes of the sides to determine if the quadrilateral is a parallelogram.

If opposite sides are parallel, the quadrilateral is a parallelogram.

- $\overline{AB}$ is opposite $\overline{DC}$.
- $\overline{BC}$ is opposite $\overline{AD}$.

\[
m_{\overline{AB}} = \frac{\Delta y}{\Delta x} = \frac{1-8}{11-0} = -\frac{7}{11}
\]
\[
m_{\overline{BC}} = \frac{\Delta y}{\Delta x} = \frac{-6-1}{0-11} = \frac{7}{11}
\]
\[
m_{\overline{DC}} = \frac{\Delta y}{\Delta x} = \frac{-6-1}{0-(-11)} = \frac{7}{11}
\]
\[
m_{\overline{AD}} = \frac{\Delta y}{\Delta x} = \frac{1-8}{-11-0} = \frac{7}{11}
\]

The opposite sides are parallel: $\overline{AB} \parallel \overline{DC}$ and $\overline{BC} \parallel \overline{AD}$. Therefore, the quadrilateral is a parallelogram.
3. Examine the slopes of consecutive sides to determine if the sides are perpendicular.

If the slopes of consecutive sides are opposite reciprocals of each other, then the sides intersect at right angles.

If the sides intersect at right angles, then the parallelogram is a rhombus or square.

Let’s use consecutive sides $AB$ and $BC$.

$$m_{AB} = -\frac{7}{11} \quad \text{and} \quad m_{BC} = \frac{7}{11}$$

The slopes are not opposite reciprocals, so the parallelogram is not a rectangle or a square.

4. Calculate the lengths of the sides.

In a rhombus, the sides are congruent.

Use the distance formula to calculate the lengths of the sides.

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

$$AB = \sqrt{(11 - 0)^2 + (1 - 8)^2} \quad \text{and} \quad BC = \sqrt{(0 - 11)^2 + (-6 - 1)^2}$$

$$AB = \sqrt{121 + 49} \quad \text{and} \quad BC = \sqrt{121 + 49}$$

$$AB = \sqrt{170} \quad \text{and} \quad BC = \sqrt{170}$$

$$DC = \sqrt{(0 - (-11))^2 + (-6 - 1)^2} \quad \text{and} \quad AD = \sqrt{(-11 - 0)^2 + (1 - 8)^2}$$

$$DC = \sqrt{121 + 49} \quad \text{and} \quad AD = \sqrt{121 + 49}$$

$$DC = \sqrt{170} \quad \text{and} \quad AD = \sqrt{170}$$

The sides are all congruent.
5. Summarize your findings.

The quadrilateral has opposite sides that are parallel and all four sides are congruent, but the sides are not perpendicular. Therefore, the quadrilateral is a parallelogram and a rhombus, but not a square.

Example 3

Quadrilateral $ABCD$ has vertices $A(-1, 2)$, $B(1, 5)$, $C(4, 3)$, and $D(2, 0)$. Using slope, distance, and/or midpoints, classify $\square ABCD$ as a rectangle, rhombus, square, trapezoid, or kite.

1. Graph the quadrilateral.
2. Calculate the slopes of the sides to determine if the quadrilateral is a parallelogram.

If opposites sides are parallel, the quadrilateral is a parallelogram.

\[ \overline{AB} \text{ is opposite } \overline{DC}, \overline{BC} \text{ is opposite } \overline{AD}. \]

\[
\begin{align*}
    m_{\overline{AB}} &= \frac{\Delta y}{\Delta x} = \frac{(5-2)}{[1-(-1)]} = \frac{3}{2} \\
    m_{\overline{BC}} &= \frac{\Delta y}{\Delta x} = \frac{(3-5)}{(4-1)} = \frac{2}{3} \\
    m_{\overline{DC}} &= \frac{\Delta y}{\Delta x} = \frac{(3-0)}{(4-2)} = \frac{3}{2} \\
    m_{\overline{AD}} &= \frac{\Delta y}{\Delta x} = \frac{(0-2)}{[2-(-1)]} = \frac{2}{3}
\end{align*}
\]

The opposite sides are parallel: \( \overline{AB} \parallel \overline{DC} \) and \( \overline{BC} \parallel \overline{AD} \). The quadrilateral is a parallelogram.

3. Examine the slopes of consecutive sides.

If consecutive sides are perpendicular, the angles at the vertices are right angles and the parallelogram is a rectangle or a square.

The slopes are opposite reciprocals; therefore, \( \overline{AB} \perp \overline{BC} \) and \( \overline{DC} \perp \overline{AD} \), and the angles at the vertices are right angles, indicating a rectangle or square.
4. Calculate the lengths of the sides.

If the sides are congruent, the parallelogram with four right angles is a square.

Use the distance formula to calculate the lengths of the sides.

\[ d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \]

\[ AB = \sqrt{1 - (-1)^2 + (5 - 2)^2} \quad BC = \sqrt{(4 - 1)^2 + (3 - 5)^2} \]
\[ AB = \sqrt{(2)^2 + (3)^2} \quad BC = \sqrt{(3)^2 + (-2)^2} \]
\[ AB = \sqrt{4 + 9} \quad BC = \sqrt{9 + 4} \]
\[ AB = \sqrt{13} \quad BC = \sqrt{13} \]

\[ DC = \sqrt{(4 - 2)^2 + (3 - 0)^2} \quad AD = \sqrt{[2 - (-1)]^2 + (0 - 2)^2} \]
\[ DC = \sqrt{(2)^2 + (3)^2} \quad AD = \sqrt{(3)^2 + (-2)^2} \]
\[ DC = \sqrt{4 + 9} \quad AD = \sqrt{9 + 4} \]
\[ DC = \sqrt{13} \quad AD = \sqrt{13} \]

The sides are all congruent.

5. Summarize your findings.

The quadrilateral has opposite sides that are parallel, four right angles, and four congruent sides, making the quadrilateral a parallelogram, a rectangle, a rhombus, and most specifically, a square.
Example 4

Use what you know about the diagonals of rectangles, rhombuses, squares, kites, and trapezoids to classify the quadrilateral given the vertices \(M(0, 3), A(5, 2), T(6, -3),\) and \(H(-1, -4)\).

1. Graph the quadrilateral.
2. Determine if the diagonals bisect each other.

If the diagonals bisect each other, then the quadrilateral is a parallelogram.

Find the midpoints of the diagonals using the midpoint formula.

Let’s start with the midpoint of the diagonal $MT$.

$$M = \left( \frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2} \right)$$  

Midpoint formula

$$M_{MT} = \left( \frac{0 + 6}{2}, \frac{3 + (-3)}{2} \right) = \left( \frac{6}{2}, \frac{0}{2} \right) = (3, 0)$$  

Substitute values for $x_1$, $x_2$, $y_1$, and $y_2$, then solve.

Now find the midpoint of the diagonal $AH$.

$$M_{AH} = \left( \frac{5 + (-1)}{2}, \frac{2 + (-4)}{2} \right) = \left( \frac{4}{2}, \frac{-2}{2} \right) = (2, -1)$$  

Substitute values for $x_1$, $x_2$, $y_1$, and $y_2$, then solve.

The midpoints are not the same, so the diagonals do not bisect each other. This rules out the quadrilateral being any type of parallelogram, including a rectangle, rhombus, or square.

3. Calculate the slopes of the diagonals.

If the diagonals are perpendicular, then the quadrilateral could be a rhombus or a kite.

$$m_{MT} = \frac{\Delta y}{\Delta x} = \frac{-3 - 3}{6 - 0} = -\frac{6}{6} = -1$$  

$$m_{AH} = \frac{\Delta y}{\Delta x} = \frac{-4 - 2}{-1 - 5} = -\frac{-6}{6} = 1$$  

The slopes are opposite reciprocals, so the diagonals are perpendicular: $MT \perp AH$. Therefore, the quadrilateral could be a rhombus or a kite. However, since we established earlier that the diagonals do not bisect each other, the quadrilateral cannot be a rhombus.
4. Calculate the lengths of the sides of the quadrilateral.

   Use the distance formula: \( d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \)

   \[
   \begin{align*}
   MA &= \sqrt{(5-0)^2 + (2-3)^2} \\
   MA &= \sqrt{25 + 1} \\
   MA &= \sqrt{26} \\
   MA &= \sqrt{26} \\
   AT &= \sqrt{(6-5)^2 + (-3-2)^2} \\
   AT &= \sqrt{1 + 25} \\
   AT &= \sqrt{26} \\
   MH &= \sqrt{(-1-0)^2 + (-4-3)^2} \\
   MH &= \sqrt{1 + 49} \\
   MH &= \sqrt{50} = 5\sqrt{2} \\
   TH &= \sqrt{(-1-6)^2 + [-4-(-3)]^2} \\
   TH &= \sqrt{(-7)^2 + (-1)^2} \\
   TH &= \sqrt{49 + 1} \\
   TH &= \sqrt{50} = 5\sqrt{2}
   \end{align*}
   \]

   The adjacent pairs of sides are congruent.

5. Summarize your findings.

   The diagonals do not bisect each other but are perpendicular. Since the diagonals do not bisect each other, the quadrilateral is not a rectangle, rhombus, or square. Since the diagonals are perpendicular and two distinct pairs of adjacent sides are congruent, the quadrilateral is a kite.
Example 5

Use what you know about the diagonals of rectangles, rhombuses, squares, kites, and trapezoids to classify the quadrilateral given vertices $P(1, 5)$, $Q(5, 2)$, $R(4, -3)$, and $S(-4, 3)$.

1. Graph the quadrilateral.
2. Determine if the diagonals bisect each other.

If the diagonals bisect each other, then the quadrilateral is a parallelogram.

Find the midpoints of the diagonals using the midpoint formula.

Let’s start with the midpoint of the diagonal \( \overline{PR} \).

\[
M_{\overline{PR}} = \left( \frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2} \right) \quad \text{Midpoint formula}
\]

\[
M_{\overline{PR}} = \left( \frac{1 + 4}{2}, \frac{5 + (-3)}{2} \right) = \left( \frac{5}{2}, \frac{1}{2} \right)
\]

Substitute values for \( x_1, x_2, y_1, \) and \( y_2 \), then solve.

Now find the midpoint of the diagonal \( \overline{QS} \).

\[
M_{\overline{QS}} = \left( \frac{5 + (-4)}{2}, \frac{2 + 3}{2} \right) = \left( \frac{1}{2}, \frac{5}{2} \right)
\]

Substitute values for \( x_1, x_2, y_1, \) and \( y_2 \), then solve.

The midpoints are not the same, so the diagonals do not bisect each other. This rules out the quadrilateral being any type of parallelogram, including a rectangle, rhombus, or square.

3. Determine the slopes of the diagonals.

If the slopes are opposite reciprocals, then the diagonals are perpendicular and the quadrilateral could be a kite.

\[
m_{\overline{PR}} = \frac{\Delta y}{\Delta x} = \frac{(-3-5)}{(4-1)} = \frac{-8}{3}
\]

\[
m_{\overline{QS}} = \frac{\Delta y}{\Delta x} = \frac{(3-2)}{(-4-5)} = \frac{-1}{9}
\]

The slopes are not opposite reciprocals. Therefore, the diagonals are not perpendicular. The quadrilateral is neither a parallelogram nor a kite.
4. Determine the lengths of the diagonals.

If the diagonals are congruent, the quadrilateral could be an isosceles trapezoid.

Use the distance formula, $d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$, to determine if the diagonals are congruent.

$$
PR = \sqrt{(4-1)^2 + (-3-5)^2} = \sqrt{9 + 164} = \sqrt{173} \\
QS = \sqrt{(-4-5)^2 + (3-2)^2} = \sqrt{81 + 1} = \sqrt{82}
$$

The diagonals are not congruent. Therefore, the quadrilateral is not an isosceles trapezoid.

5. Calculate the slopes of opposite pairs of sides.

If one pair of opposite sides is parallel, then the quadrilateral is a trapezoid.

$$
m_{PS} = \frac{\Delta y}{\Delta x} = \frac{(3-5)}{(4-1)} = \frac{-2}{5} \\
m_{PQ} = \frac{\Delta y}{\Delta x} = \frac{(2-5)}{(5-1)} = \frac{-3}{4} \\
m_{QR} = \frac{\Delta y}{\Delta x} = \frac{(-3-2)}{(4-5)} = \frac{-5}{-1} = 5 \\
m_{SR} = \frac{\Delta y}{\Delta x} = \frac{(-3-3)}{(4-(-4))} = \frac{-6}{8} = \frac{-3}{4}
$$

One pair of opposite sides, $\overline{PQ}$ and $\overline{SR}$, is parallel since the slopes are equal: $\overline{PQ} \parallel \overline{SR}$. Therefore, the quadrilateral is a trapezoid.

6. Summarize your findings.

The diagonals do not bisect each other, ruling out the quadrilateral being a parallelogram; therefore, it cannot be a rectangle, rhombus, or square. The diagonals are not perpendicular, ruling out a kite. However, one pair of opposite sides is parallel, indicating that the quadrilateral is a trapezoid. The diagonals are not congruent. Therefore, the quadrilateral is not an isosceles trapezoid.
Problem-Based Task 1.10.2: Where Are the Catchers?

Aric, Carl, Bree, and Daisy play Little League baseball. At catching practice, the coach positions each player according to his or her skill level. The four players’ positions form a rhombus. Aric, Bree, and Carl can be represented on a coordinate plane by points $A (0, 0)$, $B (-5, 0)$, and $C (-1, 3)$. Find Daisy’s location, point $D$.

Elian and Faith are the best catchers on the team, so the coach has them practice separately. The coach places Elian in a spot represented by point $E (7, -6)$ on a coordinate plane. The coach places Faith so that she’s as far away from Elian as Bree is from Daisy. If Bree, Daisy, Elian, and Faith’s positions form a rectangle, find point $F$, Faith’s position. Graph the positions of all the players.
Problem-Based Task 1.10.2: Where Are the Catchers?

Coaching

a. What does the graph of the given points look like?

b. What do you know about the diagonals of a rhombus?

c. What is the slope of $CA$?

d. What is the slope of $BD$ and why?

e. What is the equation of the line that encompasses $BD$?

f. What other line intersects $BD$ where we know the slope and one point?

g. What is the slope of $AD$, and what given point is on that line?

h. What is the equation of $AD$?

i. How can you use these two equations to find point $D$?

j. What is point $D$?

k. What point is still unknown?

l. What do we know about the slopes of the sides of a rectangle?

m. What is the slope of $BD$?

n. What is the slope of $EF$?

o. What is the equation of $EF$?

p. What other line intersects $EF$ where we know the slope and one point?

q. What is the equation of this line?

r. How can you use the two equations to find point $F$?

s. What is point $F$?

t. What is the graph of the final positions of all the catchers?
Problem-Based Task 1.10.2: Where Are the Catchers?

Coaching Sample Responses

a. What does the graph of the given points look like?

b. What do you know about the diagonals of a rhombus?

They intersect at right angles, so the lines are perpendicular, and the slopes are opposite reciprocals.

c. What is the slope of \( \overline{CA} \)?

\[
m_{\overline{CA}} = \frac{\Delta y}{\Delta x} = \frac{0 - 3}{0 - (-1)} = \frac{-3}{1} = -3
\]

d. What is the slope of \( \overline{BD} \) and why?

\[
m_{\overline{BD}} = \frac{1}{3} \text{ because it is the opposite reciprocal of } \overline{CA}, \text{ since } \overline{CA} \perp \overline{BD}.
\]
e. What is the equation of the line that encompasses $BD$?

We know that point $B(-5, 0)$ is on the line and the slope is $\frac{1}{3}$. Use the slope-intercept form of the equation to find the equation of the line.

\[
y = mx + b \\
0 = \frac{1}{3}(-5) + b \\
0 = -\frac{5}{3} + b \\
b = \frac{5}{3}
\]

f. What other line intersects $BD$ where we know the slope and one point?

There are two other lines: $CD$ and $AD$.

g. What is the slope of $AD$, and what given point is on that line?

$AD$ is parallel to $BC$ because opposite pairs of sides in a parallelogram are parallel and a rhombus is a parallelogram. When lines are parallel, their slopes are equal.

\[
m_{BC} = \frac{\Delta y}{\Delta x} = \frac{3-0}{-1-(-5)} = \frac{3}{4} \\
m_{AD} = \frac{3}{4}
\]

Point $A$ is on the line and is given, $A(0, 0)$.

h. What is the equation of $AD$?

Insert the slope of $AD$ and the point $A(0, 0)$ into the equation $y = mx + b$.

\[
y = mx + b \\
0 = \frac{3}{4}(0) + b \\
b = 0 \\
y = -\frac{x}{4}
\]
i. How can you use these two equations to find point \( D \)?

Solve the two equations as a system of equations. The solution will be the intersection point of the two lines, point \( D \).

j. What is point \( D \)?

Set the two equations equal to each other since they are both equal to \( y \).

\[
\frac{1}{3}x + \frac{5}{3} = \frac{3}{4}x
\]

Find \( x \).

\[
\begin{align*}
\frac{1}{3}x &= \frac{5}{3} - \frac{3}{4}x \\
\frac{12}{3}x &= 4 - 9 \\
x &= \frac{5}{3} \cdot \frac{12}{5} \\
x &= \frac{60}{15} \\
x &= 4
\end{align*}
\]

Use \( x \) to find \( y \).

\[
\begin{align*}
y &= -\frac{3}{4}x \\
y &= -\frac{3}{4}(4) \\
y &= 3
\end{align*}
\]

Point \( D \) is at \((4, 3)\).
k. What point is still unknown?
Point $F$ is the second unknown point.

l. What do we know about the slopes of the sides of a rectangle?
The opposite sides are parallel and consecutive sides are perpendicular. The opposite sides have the same slope and the consecutive sides have slopes that are opposite reciprocals.

m. What is the slope of $BD$?
From part c, we know $m_{BD} = \frac{1}{3}$.

n. What is the slope of $EF$?
$EF$ is parallel to $BD$, so the slopes are equal.
$$m_{EF} = \frac{1}{3}$$

o. What is the equation of $EF$?
We can determine the equation of the line if we know the slope and a point on the line.
We know the slope, $\frac{1}{3}$, and a point on the line, $E(7, -6)$. Substitute these into $y = mx + b$.
$$y = mx + b$$
$$y = \frac{1}{3}x - \frac{25}{3}$$

p. What other line intersects $EF$ where we know the slope and one point?
$BF$ intersects $EF$. 
q. What is the equation of the line?

We know the slope of \( BF \) is perpendicular to \( EF \). Therefore, the slopes are opposite reciprocals. The slope of \( EF \) is \( \frac{1}{3} \). This means the slope of \( BF \) is \(-3\). We know point \( B \) \((-5, 0)\).

Substitute this information into the slope-intercept form of the equation.

\[
y = mx + b
\]

\[
0 = -3(-5) + b
\]

\[
0 = 15 + b
\]

\[
b = -15
\]

r. How can you use the two equations to find point \( F \)?

Solve the two equations as a system. The solution will be the intersection point, \( F \).

s. What is point \( F \)?

Set the equations equal to each other since they are both equal to \( y \).

\[
\frac{1}{3} x - \frac{25}{3} = -3x - 15
\]

Find \( x \).

\[
\frac{1}{3} x = -15 + \frac{25}{3}
\]

\[
\frac{1}{3} x = -15 + 8
\]

\[
\frac{1}{3} x = 1
\]

\[
x = 3
\]

Point \( F \) is at \((-2, -9)\).
t. What is the graph of the final positions of all the catchers?

Recommended Closure Activity
Select one or more of the essential questions for a class discussion or as a journal entry prompt.
Practice 1.10.2: Proving Properties of Special Quadrilaterals

For problems 1–8, use the given coordinates as well as slope, distance, midpoints, and/or diagonals to classify each quadrilateral in as many ways as possible (parallelogram, rectangle, rhombus, square, kite, trapezoid, or isosceles trapezoid). Justify your answers.

1. \(A(-5, 6), B(3, -3), C(0, -6), D(-9, 3)\)

2. \(E(0, 2), F(4, 2), G(4, -2), H(-1, -3)\)

3. \(I(-6, 7), J(-3, 4), K(-6, 1), L(-9, 4)\)

4. \(M(-3, 8), N(2, 5), O(-1, 0), P(-6, 3)\)

5. \(P(1, 5), Q(5, 2), R(3, -1), S(-1, 2)\)

6. \(T(-6, -4), U(6, -4), V(3, -8), W(-3, -8)\)

7. \(W(3, 3), X(8, 1), Y(4, -9), Z(-1, -7)\)

8. \(A(2, -2), B(9, -2), C(9, -9), D(2, -9)\)

Use the information given in each problem that follows to write proofs.

9. Given that quadrilateral \(ABCD\) below is a rhombus, prove that the diagonals form four congruent triangles.

10. Given the vertices of a rhombus, \(A(0, 0), B(b, c), \quad C\left(b + \sqrt{b^2 + c^2}, c\right)\), and \(D\left(\sqrt{b^2 + c^2}, 0\right)\), prove that the diagonals form four right angles. (\textit{Hint}: Think about the product of the slopes of perpendicular lines.)
Practice 1.1.1: Investigating Properties of Parallelism and the Center, pp. 14–18

1. The triangle has been dilated. The corresponding sides of the triangle are parallel and the scale factor is consistent \( k = 1/2 \). Additionally, the preimage points and image points are collinear with the center of dilation.

2. The quadrilateral has not been dilated. The scale factor is inconsistent between corresponding sides.

3. The rectangle has not been dilated. The scale factor is inconsistent between corresponding sides.

4. The rectangle has been dilated. The corresponding sides of the rectangle are parallel and the scale factor is consistent \( k = 1.6 \). Additionally, the preimage points and image points are collinear with the center of dilation.

5. \( k = 1/3 \); reduction

6. \( k = 0.4 \); reduction

7. \( k = 1 \); congruency transformation

8. \( k = 1.25 \); enlargement; the scale factor is greater than 1; therefore, it is an enlargement.

9. No, because the scale factors of corresponding sides are inconsistent.

10. \( k = 1.5 \); enlargement; the scale factor is greater than 1; therefore, it is an enlargement.

Practice 1.1.2: Investigating Scale Factors, p. 24

1. 10.125

2. 28.5

3. 4.6

4. 3/5

5. \( T'(\ -3, \ -1), \ U'(\ -2, \ -2), \ V'(\ -2/3, \ -1) \)

6. \( B'(\ -2, \ 0), \ D'(\ -10, \ -12), \ E'(\ 6, \ -8) \)

7. \( N'(\ -9.6, \ -3.2), \ O'(\ 4.8, \ 8), \ P'(\ 6.4, \ -12.8) \)

8. \( E'(\ 1.2, \ 2.7), \ F'(\ 1.5, \ 0.9), \ G'(\ 2.7, \ 3) \)

9. \( I''(\ 3.375, \ 2.8125), \ J''(\ 1.125, \ 1.125), \ K''(\ -1.6875, \ 2.25) \)

\[ k = 9/16 \text{ or } 0.5625 \]

10. 6 feet by 8 feet

Practice 1.2.1: Copying Segments and Angles, pp. 42–43

1–10. Check students’ work for accuracy.

Practice 1.2.2: Bisecting Segments and Angles, pp. 59–60

1–10. Check students’ work for accuracy.

Practice 1.2.3: Constructing Perpendicular and Parallel Lines, p. 78

1–10. Check students’ work for accuracy.

Practice 1.3.1: Constructing Equilateral Triangles Inscribed in Circles, pp. 96–97

1–3. Check students’ work for accuracy. Be sure each of the vertices lies on the circle.

4. Check students’ work for accuracy. Be sure each of the vertices lies on the circle and the radius of the circle is twice the length of the given segment.

5. Check students’ work for accuracy. Be sure each of the vertices lies on the circle and the radius of the circle is equal to half the length of the given segment.

6–8. Check students’ work for accuracy. Be sure each of the vertices lies on the circle.

9. Check students’ work for accuracy. Be sure each of the vertices lies on the circle and the radius of the circle is twice the length of the given segment.

10. Check students’ work for accuracy. Be sure each of the vertices lies on the circle and the radius of the circle is equal to half the length of the given segment.

Practice 1.3.2: Constructing Squares Inscribed in Circles, p. 108

1–2. Check students’ work for accuracy. Be sure each of the vertices lies on the circle.

3–6. Check students’ work for accuracy. Be sure each of the vertices lies on the circle and the radius of the circle is equal to the length of the given segment.

7. Check students’ work for accuracy. Be sure each of the vertices lies on the circle and the radius of the circle is equal to twice the length of the given segment.

8. Check students’ work for accuracy. Be sure each of the vertices lies on the circle and the radius of the circle is equal to half the length of the given segment.

9–10. Check students’ work for accuracy. Be sure each of the vertices lies on the circle and the radius of the circle is equal to half the length of the given segment.

Practice 1.3.3: Constructing Regular Hexagons Inscribed in Circles, pp. 123–124

1. Check students’ work for accuracy. Be sure each of the vertices lies on the circle.

2–3. Check students’ work for accuracy. Be sure each of the vertices lies on the circle and the radius of the circle is equal to the length of the given segment.

4. Check students’ work for accuracy. Be sure each of the vertices lies on the circle and the radius of the circle is equal to twice the length of the given segment.
5. Check students’ work for accuracy. Be sure each of the vertices lies on the circle and the radius of the circle is equal to half the length of the given segment.

6. Check students’ work for accuracy. Be sure each of the vertices lies on the circle.

7–8. Check students’ work for accuracy. Be sure each of the vertices lies on the circle and the radius of the circle is equal to the length of the given segment.

9. Check students’ work for accuracy. Be sure each of the vertices lies on the circle and the radius of the circle is equal to twice the length of the given segment.

10. Check students’ work for accuracy. Be sure each of the vertices lies on the circle and the radius of the circle is equal to half the length of the given segment.

Practice 1.4.1: Describing Rigid Motions and Predicting the Effects, pp. 141–145

1. Rotation; the orientation changed, but the images are not mirror reflections of each other.

2. Translation; 5 units left and 3 units up; the orientation stayed the same.

3. Reflection; the line of reflection is $y = -3$; the orientation changed and the preimage and image are mirror reflections of each other; the line of reflection is the perpendicular bisector of the segments connecting the vertices of the preimage and image.

4. [Diagram of a triangle with vertices labeled A, B, and C, and their images A', B', and C'.]

5. [Diagram of a triangle with vertices labeled A, B, C, and their images A', B', C'.]

6. [Diagram of a triangle with vertices labeled A, B, C, and their images A', B', C'.]

7. [Diagram of a pentagon with a vertical line of reflection.]
8. Answers may vary. Sample answer: Translate both chairs 2 units to the right.

9. Answers may vary. Sample answer: Translate both chairs 2 units to the right.

10. Answers may vary. Sample answer: First, reflect the hexagon over a horizontal line just below the solar panel. Then translate the reflected hexagon about 1 unit to the right and 1 unit down.

Practice 1.4.2: Defining Congruence in Terms of Rigid Motions, pp. 157–161

1. Congruent; a translation occurred 7 units to the left and 3 units up. Translations are rigid motions.
2. Congruent; a rotation has occurred. Rotations are rigid motions.
3. Not congruent; a vertical compression has occurred with a scale factor of 1/2. Compressions are non-rigid motions.
4. Not congruent; a horizontal compression has occurred with a scale factor of 2/3. Compressions are non-rigid motions.
5. Congruent; a reflection has occurred. Reflections are rigid motions.
6. Not congruent; a vertical stretch has occurred with a scale factor of 5/3. Stretches are non-rigid motions.
7. The outer triangle has been dilated by a scale factor of 2/3. Since dilations are non-rigid motions, the triangles are not congruent.
8. The target has undergone a rotation. Since rotations are rigid motions, the targets are congruent.
9. The art is a reflection. Since reflections are rigid motions, the A+ on top is congruent to the A+ on the bottom.
10. Answers may vary. Sample answer: The windowpanes are congruent. Pane 1 can be translated to the right 2 units to create pane 2. Pane 1 can be translated 4 units to the right to create pane 3. Panes 1, 2, and 3 can be reflected over the horizontal line passing through the bottom of the panes to create panes 4, 5, and 6. Since all the transformations described are rigid motions, the panes are congruent.
Practice 1.5.1: Triangle Congruency, pp. 172–175
1. Sample answer: \(\triangle NLM \cong \triangle HJJ\)
2. Sample answer: \(\triangle PQR \cong \triangle TSV\)
3. Sample answer: \(\triangle USW \cong \triangle LNO\)
4. \(\angle H \cong \angle M\), \(\angle I \cong \angle N\), \(\angle J \cong \angle P\), \(\overline{HI} \cong \overline{MN}\), \(\overline{IJ} \cong \overline{NP}\), \(\overline{HJ} \cong \overline{MP}\)
5. \(\angle B \cong \angle H\), \(\angle D \cong \angle I\), \(\angle E \cong \angle L\), \(\overline{BD} \cong \overline{HI}\), \(\overline{DE} \cong \overline{JI}\), \(\overline{BE} \cong \overline{HL}\)
6. \(\angle N \cong \angle T\), \(\angle P \cong \angle V\), \(\angle R \cong \angle X\), \(\overline{NP} \cong \overline{TV}\), \(\overline{PR} \cong \overline{VX}\), \(\overline{NR} \cong \overline{TX}\)
7. Yes, the triangles are congruent; \(\triangle DGA \cong \triangle IJJ\)
8. Yes, the triangles are congruent; \(\triangle MNP \cong \triangle RTS\)
9. Yes, the triangles are congruent; \(\triangle BCD \cong \triangle FGE\)
10. No, the triangles are not congruent.

Practice 1.5.2: Explaining ASA, SAS, and SSS, pp. 185–187
1. SAS
2. ASA
3. Congruency cannot be determined; the identified congruent parts form SSA, which is not a triangle congruence statement.
4. ASA
5. Congruency cannot be determined; the identified congruent parts form SSA, which is not a triangle congruence statement.
6. SSS
7. \(\triangle DEF \cong \triangle TVS\); SAS
8. There is not enough information to determine if the pieces of wood are congruent. The information provided follows AAA, which is not a triangle congruence statement.
9. The pieces of fabric are congruent; SAS
10. The pieces of fabric are congruent; SAS or SSS

Practice 1.6.1: Defining Similarity, pp. 200–204
1. \(\angle C = 44^\circ\), \(\angle D = \angle E = 68^\circ\), \(CB = 4.5\), \(DF = 2.1\)
2. \(\angle L = \angle M = \angle P = 45^\circ\), \(MP = 9\), \(NP = 6\)
3. The triangles are similar. \(\triangle ABC\) can be dilated by a scale factor of 2 with the center at (0, 0) to obtain \(\triangle DEF\).
4. The triangles are similar. \(\triangle ABC\) can be dilated by a scale factor of 1 with center at (0, 0) and then translated 3 units vertically and 2 units horizontally to obtain \(\triangle DEF\).
5. Similarity transformations preserve angle measures. The triangles are not similar because the angle measures in each triangle are different.
6. The triangles are similar. \(\triangle ABC\) can be dilated by a scale factor of 2.5 with the center at (0, 0) and then translated 3 units vertically and 5 units horizontally to obtain \(\triangle DEF\).
7. The triangles are similar. \(\triangle ABC\) can be dilated by a scale factor of 2/7 with the center at (0, 0) and then rotated 180˚ clockwise about the origin to obtain \(\triangle DEF\).
8. The triangles are similar. \(\triangle ABC\) can be dilated by a scale factor of 3/2 with the center at (0, 0) and then reflected over the line \(x = 0\) to obtain \(\triangle DEF\).
9. Similarity transformations preserve angle measures. The triangles are not similar because the angle measures in each triangle are different.
10. The triangles are similar. \(\triangle ABC\) can be dilated by a scale factor of 5 with the center at (0, 0) and then rotated 90˚ counterclockwise about the origin to obtain \(\triangle DEF\).

1. There is not enough information to determine similarity.
2. Yes, \(\triangle ABC \sim \triangle XYZ\) because of the AA Similarity Statement.
3. Yes, \(\triangle ABC \sim \triangle XYZ\) because of the AA Similarity Statement.
4. \(\triangle ABC \sim \triangle XYZ\); \(x = 3 1/3\)
5. \(\triangle ABC \sim \triangle XYZ\); \(x = 2\)
6. \(\triangle BCD \sim \triangle BDA\); \(x = 10 2/3\)
7. 3 feet
8. 9 feet
9. 2.16 meters
10. 60 feet

Practice 1.7.1: Proving Triangle Similarity Using Side-Angle-Side (SAS) and Side-Side-Side (SSS) Similarity, pp. 226–229
1. \(\triangle ABC \sim \triangle DEF\) by SSS
2. \(\triangle ACE \sim \triangle BCD\) by SAS
3. \(\triangle ABC \sim \triangle DEF\) by SSS
4. not similar; corresponding sides are not proportional
5. \(\triangle ABC \sim \triangle EFD\) ; SAS
6. not similar; corresponding sides are not proportional
7. \(\triangle ABC \sim \triangle DEF\) ; SSS
8. \(x = 7\)
9. \(x = 5\)
10. \(x = 6\)

Practice 1.7.2: Working with Ratio Segments, pp. 239–242
1. \(CD = 12 3/8\) units
2. \(BC = 3 1/3\) units
3. \(DE = 8.4\) units
4. \(CD = 6\) units
5. \(BC = 10\) units; \(CD = 12\) units
6. \(CB = 21\) units; \(CD = 35\) units
7. Yes; the sides are proportional.
8. No; the sides are not proportional.
9. No; the sides are not proportional.
10. Yes; the sides are proportional.

Practice 1.7.3: Proving the Pythagorean Theorem Using Similarity, pp. 252–256
1. \(x = 4\sqrt{2} = 5.7\) units
2. \(x = \frac{3\sqrt{26}}{2} = 7.6\) units
3. \( x = \sqrt{3} = 1.73 \) units

4. \( x = \frac{5\sqrt{6}}{2} = 19.4 \) units

5. \( a = 2\sqrt{6} = 4.9 \) units; \( c = \frac{1}{6}; f = 4 \frac{4}{5} \)

6. \( a = 600; b = 175; \), \( x = 168 \)

7. \( c = 4\sqrt{2} = 5.7 \) units; \( e = f = 2\sqrt{2} = 2.8 \) units

8. \( a = \sqrt{6} = 2.4 \) units; \( b = \sqrt{3} = 1.7 \) units; \( x = \sqrt{2} = 1.4 \) units

9. \( e = 3.6; f = 6.4 \)

10. A segment, \( \overline{DE} \), drawn on line \( e \) is equal in length to \( a \). Locate a point \( F \) on a line \( m \) equal in length to \( b \), which is perpendicular to \( \overline{DE} \). Connect points \( F \) and \( E \). Call this segment \( x \). \( \triangle FED \) is a right triangle, so \( a^2 + b^2 = x^2 \). It is also true that \( a^2 + b^2 = c^2 \), so \( x^2 = c^2 \), or \( x = c \). \( \triangle ABC \equiv \triangle FED \) by the Side-Side-Side Congruence Statement. For this reason, \( \angle C \equiv \angle D \) and \( \angle C \) must be a right angle, making \( \triangle ABC \) a right triangle.

**Practice 1.7.4: Solving Problems Using Similarity and Congruence, pp. 266–271**

1. 33 ft
2. 8.75 ft
3. 58 1/3 ft
4. 649.6 m
5. 23 m
6. 3.2 ft
7. 33.8 ft
8. 22.3 m
9. 7.8 m
10. 30 m

**Practice 1.8.1: Proving the Vertical Angles Theorem, pp. 292–295**

1. Answers may vary. Sample answer: Adjacent angles are \( \angle 2 \) and \( \angle 3 \) as well as \( \angle 3 \) and \( \angle 4 \). Nonadjacent angles are \( \angle 1 \) and \( \angle 4 \) as well as \( \angle 5 \) and \( \angle 2 \).

2. Supplementary angles are \( \angle 7 \), \( \angle 1 \), and \( \angle 2 \). Statement: \( m\angle 7 + m\angle 1 + m\angle 2 = 180 \).

3. \( \angle 1 \) and \( \angle 4 \) are vertical angles. Statement: \( \angle 1 \equiv \angle 4 \).

4. \( \angle 4 \) and \( \angle 5 \) are complementary angles. Statement: \( m\angle 4 + m\angle 5 = 90 \).

5. 131°
6. 79°
7. 102°
8. 48°
9. Since \( \overline{AB} \perp \overline{CD} \) as this was given, \( \angle 1 \) and \( \angle 2 \), \( \angle 2 \) and \( \angle 3 \), and \( \angle 1 \) and \( \angle 4 \) all form linear pairs. This means that those pairs of angles are also supplementary by the Supplement Theorem. Therefore, \( m\angle 1 + m\angle 2 = 180 \), \( m\angle 2 + m\angle 3 = 180 \), and \( m\angle 1 + m\angle 4 = 180 \). Given that \( \angle 1 \) is a right angle, by the definition of right angles, \( m\angle 1 = 90 \). Use substitution so that \( 90 + m\angle 2 = 180 \), and by the Subtraction Property, \( m\angle 2 = 90 \). By definition, \( \angle 2 \) is a right angle. Use substitution so that \( 90 + m\angle 3 = 180 \), and by the Subtraction Property \( m\angle 3 = 90 \). By definition, \( \angle 3 \) is a right angle. Use substitution so that \( 90 + m\angle 4 = 180 \), and by the Subtraction Property \( m\angle 4 = 90 \). By definition, \( \angle 4 \) is a right angle. Therefore, \( \angle 2 \), \( \angle 3 \), and \( \angle 4 \) are right angles.

10. \( x = 13 \)

**Practice 1.8.2: Proving Theorems About Angles in Parallel Lines Cut by a Transversal, pp. 312–316**

1. \( 88^\circ \), because alternate interior angles in a set of parallel lines intersected by a transversal are congruent.
2. \( 90^\circ \), because same-side interior angles in a set of lines intersected by a transversal are supplementary.
3. \( 86^\circ \), because alternate interior angles in a set of lines intersected by a transversal are congruent.
4. \( 57^\circ \), because same-side exterior angles in a set of parallel lines intersected by a transversal are supplementary.
5. \( 144^\circ \), because corresponding angles in a set of lines intersected by a transversal are congruent.
6. \( 111^\circ \)
7. \( 95^\circ \)
8. \( m\angle 1 = 100 \), \( m\angle 2 = m\angle 3 = 80 \), \( x = 11 \), and \( y = 12 \)

<table>
<thead>
<tr>
<th>Statements</th>
<th>Reasons</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. ( |m ) and ( e ) is the transversal.</td>
<td>1. Given</td>
</tr>
<tr>
<td>2. ( \angle 2 \equiv \angle 6 )</td>
<td>2. Corresponding Angles Postulate</td>
</tr>
<tr>
<td>3. ( \angle 6 ) and ( \angle 8 ) are a linear pair.</td>
<td>3. Definition of a linear pair</td>
</tr>
<tr>
<td>4. ( \angle 6 ) and ( \angle 8 ) are supplementary.</td>
<td>4. If two angles form a linear pair, then they are supplementary.</td>
</tr>
<tr>
<td>5. ( m\angle 6 + m\angle 8 = 180 )</td>
<td>5. Supplement Theorem</td>
</tr>
<tr>
<td>6. ( m\angle 2 + m\angle 8 = 180 )</td>
<td>6. Substitution</td>
</tr>
<tr>
<td>7. ( \angle 2 ) and ( \angle 8 ) are supplementary.</td>
<td>7. Supplement Theorem</td>
</tr>
<tr>
<td>8. Since ( \ell ) is a transversal and ( \ell \perp m ), ( \angle 1 ) is a right angle. ( m\angle 1 = 90 ) because of the definition of a right angle. Since lines ( m ) and ( n ) are parallel, ( \angle 1 \equiv \angle 2 ) because of the Corresponding Angles Postulate. ( m\angle 1 = m\angle 2 ) because of the definition of congruent angles. ( m\angle 2 = 90 ) by substitution. ( \angle 2 ) is a right angle. Therefore, ( \ell \perp m ) because of the definition of perpendicular lines.</td>
<td></td>
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</table>

**Practice 1.9.1: Proving the Interior Angle Sum Theorem, pp. 333–336**

1. \( m\angle B = 20 \)
2. \( m\angle B = 56 \)
3. \( m\angle B = 81; m\angle C = 21 \)
4. \( m\angle A = 85; m\angle B = 52; m\angle C = 43 \)
5. \( m\angle A = 92; m\angle B = 29 \)
6. \( m\angle A = 128; m\angle B = 52 \)
7. \( m\angle CAB = 26; m\angle ABC = 24 \)
8. \( m\angle CAB = 25; m\angle ABC = 65 \)
9. \( m\angle CAB = 10; m\angle ABC = 125 \)
10. \( \triangle ABC \) is a triangle. By the Triangle Sum Theorem, the sum of the measures of \( \angle 1, \angle 2, \) and \( \angle 3 \) is equal to 180°. If two angles form a linear pair, they are supplementary, so \( \angle 3 \) and \( \angle 4 \) are also supplementary. The sum of the measures of \( \angle 3 \) and \( \angle 4 \) is equal to the sum of the measures of \( \angle 1, \angle 2, \) and \( \angle 3 \) by substitution. The measure of \( \angle 4 \) is equal to the sum of the measures of \( \angle 1 \) and \( \angle 2 \) by the Subtraction Property.

**Practice 1.9.2: Proving Theorems About Isosceles Triangles**, pp. 350–353

1. \( m\angle A = 40; m\angle C = 70 \)
2. \( m\angle ADB = 108; m\angle DCB = 72; m\angle DBC = 36 \)
3. \( m\angle A = 120; m\angle B = m\angle C = 30 \)
4. \( m\angle A = m\angle C = 45; m\angle B = 90 \)
5. \( x = 3 \)
6. \( x = 78; y = 51 \)
7. \( x = 13 \)
8. \( \triangle ABC \) is not isosceles.
9. \( \triangle ABC \) is isosceles; \( \overline{AB} \equiv \overline{BC} \).
10. Given \( \triangle ABC \) is equiangual, prove \( \triangle ABC \) is equilateral.

Since \( \triangle ABC \) is equiangual, \( \angle A \equiv \angle B \) and \( \overline{AB} \equiv \overline{BC} \) by the converse of the Isosceles Triangle Theorem, \( \overline{AB} \equiv \overline{BC} \) and \( \overline{AC} \equiv \overline{BC} \), so by the Transitive Property, \( \overline{AB} \equiv \overline{BC} \equiv \overline{AC} \). Therefore, \( \triangle ABC \) is equilateral.

**Practice 1.9.3: Proving the Midsegment of a Triangle**, pp. 372–378

1. \( BC = 12; XZ = 7.5; m\angle BZX = 55 \)
2. \( BC = 21; YZ = 5.625; m\angle AXY = 72 \)
3. \( XY = 5 \)
4. \( AB = 42 \)
5. \((–9, –2), (5, 8), (5, –2)\)
6. \((2, 6), (4, 3), (8, 5)\)
7. Use the slope formula to show that \( \overline{EF} \) and \( \overline{BC} \) have the same slope equal to \( \frac{3}{4} \). Therefore, \( \overline{EF} \parallel \overline{BC} \).

Use the distance formula to find \( EF = 5 \) and \( BC = 10 \).

So, \( EF = \frac{1}{2} BC \).

8. Use the slope formula to show that \( \overline{EF} \) and \( \overline{BC} \) have the same slope equal to \(-1\). Therefore, \( \overline{EF} \parallel \overline{BC} \). Use the distance formula to find \( EF = \sqrt{2} \) and \( BC = 2\sqrt{2} \).

So, \( EF = \frac{1}{2} BC \).

9. The midpoint of \( \overline{AC} = (1, 1) \); the midpoint of \( \overline{BC} = (4, 2) \); use the slope formula to show that \( \overline{EF} \) and \( \overline{AB} \) have the same slope equal to \( \frac{1}{3} \). Therefore, \( \overline{EF} \parallel \overline{AB} \). Use the distance formula to find \( EF = \sqrt{10} \) and \( AB = 2\sqrt{10} \). So, \( EF = \frac{1}{2} AB \).

10. The midpoint of \( \overline{AC} = (1, 1) \); the midpoint of \( \overline{BC} = (4, 2) \); use the slope formula to show that \( \overline{EF} \) and \( \overline{AB} \) have the same slope equal to \( \frac{5}{2} \). Therefore, \( \overline{EF} \parallel \overline{AB} \). Use the distance formula to find \( EF = \frac{29\sqrt{2}}{2} \) and \( AB = 2\sqrt{29} \). So, \( EF = \frac{1}{2} AB \).

**Practice 1.9.4: Proving Centers of Triangles**, pp. 406–408

1. \((-1, –3)\) is the circumcenter of \( \triangle ABC \) because the distance from this point to each of the vertices is \( \sqrt{10} \).
2. \((-2, 1)\) is the orthocenter of \( \triangle ABC \) because this point is a solution to the equation of each altitude: \( x = -2 \) and \( y = 1 \).
3. \( \triangle ABC \) is a right triangle. Therefore, \( \overline{BC} \) is the hypotenuse.
4. The distance from \( A \) to \( U \) is approximately 6.71 units. The distance from \( (5, –1) \) to \( A \) is 4.47 units. The distance from \( (5, –1) \) to \( B \) is approximately 3.5 units. The distance from \( (5, –1) \) to \( C \) is approximately 7.65 units. The distance from \( (5, –1) \) to \( C \) is approximately 5.10 units. The distance from \( (5, –1) \) to \( C \) is the maximum amount of space for the dog. Once the incenter is found, the dog’s leash can be staked in the ground.
Practice 1.10.1: Proving Properties of Parallelograms, pp. 428–430

1. No, it’s not a parallelogram because opposite sides are not parallel: \( m_{\overline{TV}} = 5 \), \( m_{\overline{WV}} = 2 \), \( m_{\overline{TV}} = -\frac{2}{3} \), \( m_{\overline{WV}} = \frac{1}{3} \).

2. Yes, it’s a parallelogram because opposite sides are parallel: \( m_{\overline{TV}} = 5 \) and \( m_{\overline{WV}} = 5 \).

3. Yes, it’s a parallelogram because opposite sides are congruent: \( GH = Hf = \sqrt{10} \) and \( HI = JG = \sqrt{26} \).

4. No, it’s not a parallelogram because opposite sides are not congruent: \( AB = \sqrt{13} \), \( CD = \sqrt{2} \), \( BC = \sqrt{29} \), and \( DA = 3\sqrt{2} \).

5. Yes, it’s a parallelogram because the midpoints of the diagonals are the same, indicating the diagonals bisect each other: \( M = \left( 3, -\frac{3}{2} \right) \).

6. No, the midpoints are not the same, indicating the diagonals do not bisect each other: \( M_{\overline{MO}} = (1, 1) \) and \( M_{\overline{NP}} = \left( 3 \cdot \frac{3}{2}, -\frac{2}{2} \right) \).

7. \( m\angle A = m\angle C = 165 \) and \( m\angle B = m\angle D = 15 ; x = 15 \) and \( y = 21 \).

8. \( m\angle A = m\angle C = 92 \) and \( m\angle B = m\angle D = 88 ; x = 18 \) and \( y = 24 \).

9. Given that \( AB \) is parallel to \( DE \), we can use the Alternate Interior Angles Theorem to show that \( m\angle BAH = m\angle DEH \).

10. Opposite sides are parallel: \( m_{\overline{AB}} = m_{\overline{CD}} = \frac{b}{a} \) and \( m_{\overline{BC}} = m_{\overline{DA}} = 0 \). The diagonals bisect each other because they have the same midpoint: \( M = \left( \frac{a + c}{2}, \frac{b}{2} \right) \).

Practice 1.10.2: Proving Properties of Special Quadrilaterals, pp. 453–454

1. Quadrilateral \( ABCD \) is a parallelogram, a rectangle, a rhombus, and a square. Justification: opposite sides are parallel, consecutive sides are perpendicular, the diagonals bisect each other, and all four sides are congruent.

2. Quadrilateral \( EFGH \) is a parallelogram and a rectangle. Justification: opposite sides are parallel, adjacent sides are perpendicular, the diagonals bisect each other, and not all four sides are congruent.

3. Quadrilateral \( JKLM \) is an isosceles trapezoid. Justification: one pair of opposite sides is parallel and the other pair of sides is congruent.

4. Quadrilateral \( NOPQ \) is a parallelogram. Justification: opposite sides are parallel, the diagonals are congruent and bisect each other.

5. Quadrilateral \( STUV \) is a parallelogram and a rhombus. Justification: opposite sides are parallel, adjacent sides are not perpendicular, the diagonals are perpendicular, and all four sides are congruent.

6. Quadrilateral \( WXYZ \) is a parallelogram, a rhombus, and a square. Justification: opposite sides are parallel, adjacent sides are perpendicular, the diagonals are parallel, and all four sides are congruent.

7. Quadrilateral \( ABCD \) is a kite. Justification: adjacent sides are congruent and the diagonals intersect at a right angle.

8. Quadrilateral \( FGHI \) is a parallelogram and a rectangle. Justification: opposite sides are parallel, adjacent sides are perpendicular, the diagonals bisect each other, and not all four sides are congruent.

<table>
<thead>
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<tbody>
<tr>
<td>1. Quadrilateral ( ABCD ) is a square.</td>
<td>1. Given</td>
</tr>
<tr>
<td>2. ( DP = PB ) ( \overline{AP} = PC )</td>
<td>2. The diagonals are congruent and bisect each other.</td>
</tr>
<tr>
<td>3. ( AC \perp DB )</td>
<td>3. The diagonals of a square are perpendicular.</td>
</tr>
<tr>
<td>4. ( \angle APD \equiv \angle APB \equiv \angle BPC \equiv \angle CPD ), and they are right angles.</td>
<td>4. Definition of perpendicular lines</td>
</tr>
<tr>
<td>5. ( \angle APD \equiv \angle APB \equiv \angle BPC \equiv \angle CPD )</td>
<td>5. All right angles are congruent.</td>
</tr>
<tr>
<td>6. ( \triangle APD \equiv \triangle APB \equiv \triangle CPB \equiv \triangle CPD )</td>
<td>6. SAS Congruence</td>
</tr>
</tbody>
</table>

10. Given \( \triangle APD \equiv \triangle APB \equiv \triangle CPB \equiv \triangle CPD \), Corresponding Parts of Congruent Triangles are Congruent. This means that \( AD \equiv AB \equiv CB \equiv CD \). Also by CPCTC, \( \angle PAD \equiv \angle PCB \). This means that \( BC \parallel AD \) since these are alternate interior angles. Then, by CPCTC again, \( \angle BAP \equiv \angle ZCD \). This means that \( AB \parallel CD \) because these angles are alternate interior angles. Since opposite sides are parallel and the sides are all congruent, by definition, quadrilateral \( ABCD \) is a rhombus.